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PLANE AND SPHERICAL TRIGONOMETRY

D. A. MURRAY, B. A., PH. D.

PLANE TRIGONOMETRY

FOR

COLLEGES AND SECONDARY SCHOOLS

BY

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PREFACE.

ALTHOUGH there are already many excellent text-books on trigonometry, there appears still to be room for one which shall avoid the extremes of expansion and brevity. Some of the most thorough and scholarly of these contain a great variety of matters which it is impossible to consider in the time usually assigned to this study in school and college. On the other hand, the explanations given in many other works are so meagre that the student is perplexed and bewildered by the new ideas which are so abruptly forced upon him, and the difficulties of the teacher are greatly increased. The manner of presentation adopted in this volume necessitates more *reading matter*, and, consequently, a somewhat larger number of pages than is found in many of the recent text-books on trigonometry. This has seemed unavoidable, however, for the general consensus of opinion among those with whom the author has conferred, is that it is essential to explain in some detail the principles of the science, in order that it may be clearly and intelligently understood by an elementary student.

With regard to the scope of the book, it may be said that it deals with the subjects considered in the ordinary course in plane trigonometry in colleges and secondary schools. It discusses the topics usually required for teachers' certificates, for entrance to college, and for examinations in trigonometry in the first year of the college curriculum. It treats of all the topics that one who has taken a few months' course in trigonometry may be reasonably expected to know.

Careful consideration has been given both to the early difficulties and to the possible future needs of the beginner. The book differs somewhat from other text-books on this branch of mathematics both in the arrangement and in the manner of presentation. The oldest and simplest part of trigonometry, namely, the solution of triangles and the associated practical problems, is concluded before the more general and abstract portions of the study are introduced. The first chapters of the book contain little more about trigonometric ratios and angular analysis than is sufficient to enable the beginner to understand clearly the arithmetical part of the science, and its simple practical applications. This arrangement seems to have several advantages. The subject is rendered far less strange at the beginning, and, by means of practical, concrete examples, the student becomes familiar with the trigonometric functions before proceeding to the more general treatment. His progress is thus made easier and more rapid. Teachers who prefer a wider generality of treatment at the outset, however, can select the chapters in a different order from that followed in the text.

An endeavour has been made to introduce the several topics in such a way that the pupil may have, from the very start, an intelligent idea of each step in advance, as well as of the ultimate purpose of the study. In some cases, especially in Chapter II. (the first chapter on trigonometry), care has been taken to prepare the mind of the learner for the reception of new ideas, by the preliminary solution of easy familiar exercises. Throughout the work the author has endeavoured to make each step clear, and thus to prevent the appearance of that puzzled feeling which has such a depressing influence on those entering upon a new study. On the other hand, he has sought to develop independence of mind and the power of mental initiative on the part of the student. Suggestions as to practical methods of work are frequently

introduced, and summaries are made in several places for the purpose of helping the pupil to get a better idea of the subject as a whole.

In the practical applications, marked attention has been given to the graphical method of solution, as well as to the method of computation. The former method serves as a check upon the latter, and affords practice in neat and careful drawing. What is perhaps more important, however, is that the students will thus become accustomed to a method which will be used by them in other studies, and which is often employed in practical work by engineers and others.

Logarithms are used almost at the beginning of the study as here presented. For this reason, and in order to avoid making a digression later on, an introductory chapter is devoted to a review on logarithms. Examples, simple ones as a rule, are given in the several articles. Questions and exercises suitable for practice and review on the separate chapters are placed at the end of the book instead of at the ends of the chapters. These collections will be found useful, both in the short reviews that may be required on the completion of each chapter and in the larger and more general reviews. Many of the examples have been taken from examination papers set in Great Britain and the United States.

Throughout the work there are many historical and other notes; and an historical sketch is given in the Appendix. It is believed that some knowledge of the historical development of trigonometry, and of the men of various times and races who have helped to advance the subject, will interest and stimulate those who are entering upon its study.

While writing this book, the author has received many valuable suggestions from Mr. J. A. Clark, B.S., of the Ithaca High School, and from several of his colleagues in the departments of mathematics and of engineering at Cornell University. He is indebted

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PLANE TRIGONOMETRY.

CHAPTER I.

LOGARITHMS: REVIEW OF TREATMENT IN ARITHMETIC AND ALGEBRA.

1. There is a large amount of computation necessary in the solution of some of the practical problems in trigonometry. The labour of making extensive and complicated calculations can be greatly lessened by the employment of a table of logarithms, an instrument which was invented for this very purpose by John Napier (1550–1617), Baron of Merchiston in Scotland, and described by him in 1614. From Henry Briggs (1556–1631), who was professor at Gresham College, London, and later at Oxford, this invention received modifications which made it more convenient for ordinary practical purposes.*

Every good treatise on algebra contains a chapter on logarithms. This brief introductory review is given merely for the purpose of bringing to mind the special properties of logarithms which make them readily adaptable to the saving of arithmetical work. A little preliminary practice in the use of logarithms will be of advantage to any one who intends to study trigonometry. A review of logarithms as treated in some standard algebra is strongly recommended.

---

* The logarithms in general use are known as Common logarithms or as Briggs's logarithms, in order to distinguish them from another system, which is also a modified form of Napier's system. The logarithms of this other modified system are frequently employed in mathematics, and are known as Natural logarithms, Hyperbolic logarithms, and also, but erroneously, as Napierian logarithms. See historical sketch in article *Logarithms* (Ency. Brit. 9th edition), by J. W. L. Glaisher.

2. Definition of a logarithm.

If $$a^x = N, \qquad (1)$$

then *x is the index of the power to which a must be raised in order to equal N.*

For some purposes, this idea is presented in these words: If $a^x = N$, then *x is the logarithm of N to the base a.*

The latter statement is taken as the *definition of a logarithm*, and is expressed by mathematical symbols in this manner, viz.:

$$x = \log_a N. \qquad (2)$$

Equations (1), (2), are equivalent; they are merely two different ways of stating a certain connection between the three quantities *a, x, N*. For example, the relations

$$2^3 = 8, \quad 5^4 = 625, \quad 10^{-3} = \tfrac{1}{1000} = .001,$$

may also be expressed by the equivalent logarithmic equations,

$$\log_2 8 = 3, \quad \log_5 625 = 4, \quad \log_{10} .001 = -3.$$

EXAMPLES.

1. Express the following equations in a logarithmic form:

$3^3 = 27$, $4^4 = 256$, $11^2 = 121$, $9^3 = 729$, $7^3 = 343$, $m^b = p$.

2. Express the following equations in the exponential form:

$\log_2 8 = 3$, $\log_5 625 = 4$, $\log_{10} 1000 = 3$, $\log_2 64 = 6$, $\log_a P = a$.

3. When the base is 2, what are the logarithms of 1, 2, 4, 8, 16, 32, 64, 128, 256?

4. When the base is 5, what are the logarithms of 1, 5, 25, 125, 625, 3125?

5. When the base is 10, what are the logarithms of 1, 10, 100, 1000, 10,000, 100,000, 1,000,000, .1, .01, .001, .0001, .00001, .000001?

6. When the base is 4, and the logarithms are 0, 1, 2, 3, 4, 5, what are the numbers?

7. When the base is 10, between what whole numbers do the logarithms of the following numbers lie: 8, 72, 235, 1140, 3470, .7, .04, .0035?

3. Properties of logarithms. Since a logarithm is the index of a power, it follows that the properties of logarithms must be derivable from the properties of indices; that is, from the laws

of indices. The laws of indices are as follows (a, m, n, being any finite quantities):

(1) $a^m \times a^n = a^{m+n}$.

(2) $\dfrac{a^m}{a^n} = a^{m-n}$. $\left[\dfrac{a^m}{a^m} = a^{m-m} = a^0$; also, $\dfrac{a^m}{a^m} = 1$. $\therefore a^0 = 1\right]$.

(3) $(a^m)^n = a^{mn}$. (4) $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$.

Let $M = a^m$, whence, $\log_a M = m$; (1)

and let $N = a^n$, whence, $\log_a N = n$. (2)

It follows that

$MN = a^{m+n}$; whence, $\log_a MN = m + n = \log_a M + \log_a N$. (3)

[If $P = a^p$, then $\log_a P = p$, $MNP = a^{m+n+p}$;

whence, $\log_a MNP = m + n + p = \log_a M + \log_a N + \log_a P$.]

Also, $$\frac{M}{N} = \frac{a^m}{a^n} = a^{m-n};$$

whence, $$\log_a \frac{M}{N} = m - n = \log_a M - \log_a N. \quad (4)$$

Also, $M^r = (a^m)^r = a^{mr}$; whence, $\log_a M^r = rm = r \log_a M$. (5)

Also, $$\sqrt[r]{M} = (a^m)^{\frac{1}{r}} = a^{\frac{m}{r}};$$

whence, $$\log_a \sqrt[r]{M} = \frac{1}{r} \cdot m = \frac{1}{r} \log_a M. \quad (6)$$

The results (3)–(6) state the properties, or are *the laws of logarithms*. They may be expressed in words as follows:

(1) *The logarithm of the* **product** *of any number of factors is equal to the* **sum** *of the logarithms of the factors.*

(2) *The logarithm of the* **quotient** *of two numbers is equal to the logarithm of the numerator* **diminished** *by the logarithm of the denominator.*

(3) *The logarithm of the* r**th power** *of a number is equal to* r **times** *the logarithm of the number.*

(4) *The logarithm of the* r**th root** *of a number is equal to* $\frac{1}{r}$**th** *of the logarithm of the number.*

Hence, if the logarithms (*i.e.* the exponents of powers) of numbers be used instead of the numbers themselves, then the operations of *multiplication* and *division* are replaced by those of *addition* and *subtraction*, and the operations of *raising to powers* and *extracting roots*, by those of *multiplication* and *division*.

4. Common system of logarithms. Any positive number except 1 may be chosen as the base, and to the base chosen there corresponds a set or system of logarithms. In the common or decimal system the base is 10, and, as will presently appear, this system is a very convenient one for ordinary numerical calculations.* In what follows, the base 10 is not expressed, but it is always understood that 10 is the base. *The logarithm of a number in the common system is the answer to the question: "What power of 10 is the number?"*

Since

$$1=10^0, \quad 10=10^1, \quad 100=10^2, \quad 1000=10^3, \quad 10000=10^4, \cdots,$$

it follows that

$$\log 1 = 0, \log 10 = 1, \log 100 = 2, \log 1000 = 3, \log 10000 = 4, \cdots.$$

This also shows that the logarithms of numbers

$$\left.\begin{array}{l} \text{between } 1 \text{ and } 10 \text{ lie between 0 and 1,} \\ \text{between } 10 \text{ and } 100 \text{ lie between 1 and 2,} \\ \text{between } 100 \text{ and } 1000 \text{ lie between 2 and 3, and so on.} \end{array}\right\} \quad (1)$$

For example,

$$9=10^{.95424}, \qquad 247=10^{2.39270}, \qquad 1453=10^{3.16227};$$

or $\log 9 = .95424$, $\log 247 = 2.39270$, $\log 1453 = 3.16227$.

Most logarithms are incommensurable numbers. (See Art. 9.) The decimal part of the logarithm is called the *mantissa*, the

* The base of the natural system of logarithms is an incommensurable number, which is always denoted by the letter e and is approximately equal to 2.7182818284.

integral part of the logarithm is called the *index* or *characteristic.*

The two great advantages of the common system, as will now be shown, are:

(1) *The characteristic of a logarithm can be written on mere inspection;*

(2) *The position of the decimal point in a number affects the characteristic alone, the mantissa being always the same for the same sequence of figures.*

Since $.1 = \frac{1}{10} = 10^{-1}$, $.01 = \frac{1}{100} = 10^{-2}$,

$.001 = \frac{1}{1000} = 10^{-3}$, $.0001 = \frac{1}{10000} = 10^{-4}$, etc.;

it follows that

$$\log .1 = -1,\ \log .01 = -2,\ \log .001 = -3,\ \log .0001 = -4, \text{ etc.} \quad (2)$$

From (1) and (2) comes the following **rule for finding the characteristic:**

When the number is greater than 1, the characteristic is positive and is one less *than the number of digits to the left of the decimal point; when the number is less than 1, the characteristic is negative, and is one* more *than the number of zeros between the decimal point and the first significant figure.*

***When** a change is made in the position of the decimal point in a number,* the value of the number is changed by some integral power of 10. Its logarithm is then changed by a whole number only, and, consequently, *its mantissa is not affected.* For example,

$$25.38 = 2538 \times 10^{-2}, \qquad 2538000 = 2538 \times 10^{3};$$

and hence, $\log 25.38 = \log 2538 - 2$, $\log 2538000 = \log 2538 + 3$.

Accordingly, it is necessary to put only the *mantissas of sequences of integers* in the tables.

5. Negative characteristics. In common logarithms *the mantissa is always kept positive.* Thus, for example, $\log 25380 = 4.40449$; $\log .002538 = \log \frac{2538}{1000000} = \log 2538 - \log 1000000 = 3.40449 - 6 = -3 + .40449$. (Never put -2.59551.)

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5. Find the value of x in $34^x = 19$.

Since
$$34^x = 19,$$
$$\log 34^x = \log 19,$$
$$x \log 34 = \log 19,$$
$$x = \frac{\log 19}{\log 34} = \frac{1.27875}{1.53148} = .83498, \text{ nearly.}$$

6. Find the value of (*a*) $\frac{374}{267}$, (*b*) $\frac{29.76}{315.2}$, (*c*) $\frac{4.132}{59.83}$, (*d*) $\frac{.0417}{.4231}$

7. Find the value of (*a*) $\frac{76.5 \times 83.21}{674.2}$, (*b*) $\frac{8.97 \times 6.36}{7.84}$, (*c*) $\frac{95.83 \times 76.49}{82.97}$

8. Find the value of (*a*) $\sqrt{63}$, (*b*) $\sqrt{630}$, (*c*) $\sqrt{6.3}$, (*d*) $\sqrt{.63}$, (*e*) $\sqrt{.063}$, (*f*) $\sqrt{.0063}$.

9. Find the value of $\sqrt{63.42 \times 74.95}$, $\sqrt{6.35 \times 10.87}$, $\sqrt{14.21 \times 17.29}$.

10. Find the value of $\sqrt{\frac{63.9 \times 72.11}{7.81 \times 6.95}}$, $\sqrt{\frac{31.21 \times 41.7}{11.39 \times 15.71}}$, $\sqrt{\frac{41.7 \times 85.6}{73.4 \times 97.8}}$.

11. Find the value of $2.5637^{\frac{5}{11}}$.

12. $\left(\frac{35}{113}\right)^{\frac{3}{4}}$.

13. Find x from the equations:

(*a*) $3^x = 35$, (*b*) $5^x = 70$, (*c*) $10^x = 36$,

(*d*) $10^x = 127$, (*e*) $10^x = 765$, (*f*) $10^x = 1364$.

CHAPTER II.

TRIGONOMETRIC RATIOS OF ACUTE ANGLES.

7. The name Trigonometry is derived from two Greek words which taken together mean 'I measure a triangle.'* At the present time the measurement of triangles is merely one of several branches included in the subject of trigonometry. The more elementary part of trigonometry is concerned with the calculation of straight and circular lines, angles, and areas belonging to figures on planes and spheres. It consists of two sections, viz. Plane Trigonometry and Spherical Trigonometry. Elementary trigonometry has many useful applications, for instance, in the measurement of areas, heights, and distances. An acquaintance with its simpler results is very helpful, and sometimes indispensable, in even a brief study of such sciences as astronomy, physics, and the various branches of engineering. Some modern branches of trigonometry require a knowledge of advanced algebra. Their results are used in the more advanced departments of mathematics and in other sciences. This work considers only the simpler portions of trigonometry, and shows some of its applications.

The truths of elementary trigonometry are founded upon geometry, and are obtained and extended by the help of arithmetic and algebra. A knowledge of the principal facts of plane geometry, and the ability to perform the simpler processes of algebra, are necessary on beginning the study of plane trigonometry. Instruments for measuring lines and angles, and accuracy in computation are required in making its practical applications.

8. Ratio. Measure. On entering upon the study of trigonometry it is very necessary to have clear ideas concerning the terms *ratio* and *incommensurable numbers* as explained in arithmetic and algebra, for these terms play a highly important part

* See historical sketch, p. 165.

in the subject. The study begins with an explanation of certain ratios which are used in it continually, and most of the numbers that appear in the solution of its problems are incommensurable.

If one quantity is half as great as another quantity in magnitude, it is said that the ratio of the first quantity to the second is as one to two, or one-half. This ratio is sometimes indicated thus, 1 : 2; but more usually it is written in the fractional form, $\frac{1}{2}$. In this example the magnitude of the second quantity is twice that of the first, and the ratio of the second quantity to the first is 2 : 1, or, adopting the more usual style, $\frac{2}{1}$, *i.e.* 2. *The ratio of two quantities* is simply the number which expresses the magnitude of the one when compared with the magnitude of the other. This ratio is obtained by finding *how many times* the one quantity contains the other, or by finding *what fraction* the one is of the other. It follows that a ratio is merely *a pure number*, and that it can be obtained only by comparing quantities of *the same kind*. Thus the ratio of the length 3 feet to the length 2 inches is $\frac{36}{2}$, *i.e.* 18; the ratio of the weight 2 pounds to the weight 3 pounds is $\frac{2}{3}$. But one cannot speak of the ratio of 3 weeks to 10 yards, for there is no sense in the questions: How many times does 3 weeks contain 10 yards? What fraction of 10 yards is 3 weeks?

When it is said that a line is ten inches long, this statement means that a line one inch long has been chosen for the unit of length, and that the first line contains ten of these units. Thus the *number* used in telling the length of a line is the ratio of the length of this line to the length of another line which has been chosen for the unit of length. The *measure* of any quantity, such as a length, a weight, a time, an angle, etc., is

$$\left\{\begin{array}{l}\text{the number of times the quantity contains}\\ \text{or, the fraction that the quantity is of}\end{array}\right\}$$

a certain quantity of the *same* kind which has been adopted as the unit of measurement. In other words, the measure of a quantity is the ratio of the quantity to the unit of measurement. For example, if half an inch is the unit of length, then the measure of a line 8 inches long is 16; if a foot is the unit of length, then the measure of the same line is $\frac{2}{3}$; if a second is the unit of time, then the measure of an hour is 3600; if an hour is the unit of time, *then the measure* of a second is $\frac{1}{3600}$.

If two quantities have a common unit of measurement, then their ratio is the ratio of their measures. For example, 1 pound being taken as the unit of weight, the ratio of a weight 3 pounds to a weight 7 pounds is $\frac{3}{7}$, which is also the ratio of the measures 3 and 7. In general, if a quantity P contains m units, and a quantity Q contains n units of the same kind as is used in the case of P, then the ratio

$$\frac{\text{quantity } P}{\text{quantity } Q} = \frac{m \text{ units}}{n \text{ units}} = \frac{m}{n}.$$

The last fraction $\frac{m}{n}$ is the ratio of the numbers m and n, which are the measures of the quantities P and Q respectively.

EXAMPLES.

1. What is the ratio of each of the following lengths to an inch, viz., 8 in., 2 ft., 3 ft. 6 in., 1.5 yd., 20 yd., a yd., b ft., c in.?

2. What is the ratio of each of the following lengths to a yard, viz., 6 yd., 3.75 yd., 8 ft., 2 ft. 6 in., 10 in., 5 in., a yd., b ft., c in.?

3. What is the measure of each of the following lengths, when a foot is the unit of length, viz., 1.5 mi., 17 yd., 3 yd. 2 ft., 8.5 ft., 2 ft. 6 in., 9 in., 2 in., a yd., b ft., c in.?

4. What is the measure of each of the following lengths, when 3 in. is the unit of length, viz., 2.5 yd., 1.5 ft., 8 in., a yd., b ft., c in.?

5. Express the ratio of 2.5 mi. to 10 yd.; and the ratio of $2\frac{1}{2}$ in. to $3\frac{1}{2}$ yd.

6. Compare the ratio of a foot to a yard with the ratio of a square foot to a square yard.

7. What is the unit of measurement in each of the following cases: when the measure of 2 ft. is 4, of 1 yd. is 72, of .5 in. is 4, of 2.5 ft. is .25?

N.B. *The following examples will be used again for purposes of illustration. The student is advised to draw figures neatly and accurately and to preserve the results carefully.*

8. In a right-angled triangle the base is 6 ft. and the hypotenuse 10 ft. What is the perpendicular? Calculate the following ratios, viz.:

$$\frac{\textit{perpendicular}}{\textit{hypotenuse}}, \quad \frac{\textit{base}}{\textit{hypotenuse}}, \quad \frac{\textit{perpendicular}}{\textit{base}},$$

$$\frac{\textit{base}}{\textit{perpendicular}}, \quad \frac{\textit{hypotenuse}}{\textit{base}}, \quad \frac{\textit{hypotenuse}}{\text{perpendicular}}.$$

What are these ratios in a triangle whose base is 6 in., and hypotenuse 10 in.? What are they when the base is 6 yd., and the hypotenuse 10 yd.? When the base is 6 mi., and the hypotenuse 10 mi.? When the base is 12 ft., and the hypotenuse 20 ft.? When the base is 3 in., and the hypotenuse 5 in.? Compare, if possible, the *angles* in these triangles.

9. In a right-angled triangle whose base is 35 ft. and perpendicular 12 ft., what is the hypotenuse? For this triangle calculate the ratios specified in Ex. 8. Calculate these ratios for a triangle whose base is 70 yd., and perpendicular 24 yd. Compare, if possible, the angles in these triangles.

10. Calculate these ratios for the triangle whose hypotenuse is 29 ft., and perpendicular 21 ft.; for the triangle whose hypotenuse is 2.9 in., and perpendicular 2.1 in. Compare, if possible, the angles in these triangles.

9. Incommensurable quantities. Approximations. If the side of a square is one foot in length, then the length of a diagonal of the square is $\sqrt{2}$ feet. Thus the ratio of the diagonal to the side is $\sqrt{2}$, a number which cannot be expressed as the ratio of two whole numbers. Two quantities whose ratio can be expressed by means of two integers are said to be *commensurable* the one with the other; when their ratio cannot be so expressed, the one quantity is said to be *incommensurable* with the other. For example, the diagonal of a square is incommensurable with the side, and the length of a circle with its diameter.* The quantities in the examples, Art. 8, are commensurable. Numbers such as $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{10}$ are incommensurable with unity, and their values cannot be found exactly. Their values, however, can be found to two, to three, to four, in fact, to as many places of decimals as one please. The greater the number of places of decimals, the more nearly will the calculated values represent the true values of the numbers. In other words, the values of incommensurable numbers can be found *approximately;* and *the degree of approximation* (that is, the nearness to the exact values) will depend only on the carefulness and patience of the calculator. In practical problems there frequently is occasion for the exercise of judgment as to the degree of approximation that is necessary and sufficient. For example, in calculating a length in inches in ordinary engineer-

* See Appendix, Note C.

ing work there is no need to go beyond the third place of decimals, for engineers are satisfied when a measurement is correct to within $\frac{1}{64}$ of an inch. As a rule the results obtained in *practical* problems in mathematics are only approximate and not exact. There are two reasons for this: first, the *data* obtained by actual measurement can only be approximate, however excellent the instruments used in measuring may be, and however skilled and careful is the person who does the measuring; second, most of the *numbers used* in the subsequent computations *are incommensurable.*

The examples at the end of this article are intended to bring out more clearly the idea of an approximate result. The answers are to be calculated to three places of decimals. It is advisable to compare the values calculated to three places of decimals with the values calculated to two places of decimals, and to note the difference between them. The following facts are supposed to be known and will be taken for granted.

(*a*) In a right-angled triangle the square of the measure of the hypotenuse is equal to the sum of the squares of the measures of the other two sides.

(*b*) The ratio of the length of any circle to its diameter is a number which is the same for all circles.* The exact value of this ratio is incommensurable and is always denoted by the symbol π (read *pi*).† The approximate values commonly used for π are 3.1416, 3.14159, $\frac{355}{113}$ (*i.e.* 3.1415929 ...), $\frac{22}{7}$ (*i.e.* $3.1\dot{4}2857\dot{1}$); of these values the last is the least accurate, but it is accurate enough for many practical purposes.

(*c*) The length of a circle of radius r is $2\pi r$ [by (*b*)]; and the enclosed area is πr^2.

NOTE 1. If a number be calculated to three or more places of decimals, then the closest approximation to, say, two places of decimals is obtained by leaving the number in the second place of decimals unchanged when the number in the third place is less than 5, and by increasing the number in the second place by unity when the number in the third place is greater than 5 or 5 followed by numbers; thus, *e.g.*, 3.72 for 3.724, 3.73 for 3.7261 and

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* This ratio and facts (*c*) are considered in Note C, Appendix. The reading only requires a knowledge of elementary geometry.

† This symbol is the initial letter of *periphereia*, the Greek word for circumference. Its earliest appearances to denote this ratio are in Jones's *Synopsis Palmariorum Mathesos*, London, 1706, and in the *Introductio in analysin infinitorum*, published in 1748 by Leonhard Euler (1707–1783), a native of Switzerland, who was one of the greatest mathematicians of his time.

3.7257. When the number in the third place is 5 and this is followed by zeros only, the number in the second place is unchanged if it is even, and is increased by unity if it is odd; thus, *e.g.*, 3.78 for 3.775, 3.78 for 3.785. In a series of calculations the errors made by following this rule tend to balance one another.

Note 2. A quantity measured to two places of decimals is correct to the hundredth part of the unit employed, and a quantity measured to three places is correct to the thousandth part of the unit. For example, the length of a circle of 10 feet diameter is 31.4159 . . . feet. For this length 31.416 or 31.42 may be taken; the former result differs from the true result by less than one-thousandth of a foot, the latter by less than one-hundredth.

EXAMPLES.

1. A finds the square root of 3 correctly to two places of decimals, and B to three. How much closer than A does B come to the exact value of the square root of 3?

2. A circle is 50 ft. in diameter. In calculating its length A takes 3.1416 as the ratio of the length of a circle to the diameter, B takes 3.14159, and C takes 22 : 7. What are the differences (in inches) between their results?

3. The radius of a circle is 49.95 ft. How nearly will a person come to the length of the circle if he assumes the radius to be 50 ft.? [In this and the following example take $\pi = 22 : 7$.]

4. It is known that the diameter of a certain circle does not differ from 100 ft. by more than 2 in. What will be the outside limits of the error made in calculating the area when the diameter is taken as 100 ft.?

5. Find the difference between the calculations of the numbers of revolutions per mile made by a 50-in. bicycle, for $\pi = 22 : 7$ and $\pi = 3.1416$.

6. A lot is 75 ft. by 200 ft. Find the diagonal distance across the lot correctly to within a tenth of an inch.

7. Find the height of an equilateral triangle whose side is 20 yd.

8. The side of an isosceles triangle is 40 ft. and the base is 30 ft.; find the height.

9. What is the length of the diagonal of a square whose side is 20 ft.?

10. What is the length of the side of a square whose diagonal is 20 ft.?

N. B. *The following examples will be used again for purposes of illustration. The student is advised to draw figures and to preserve the results with those of* Exs. 8, 9, 10, Art. 8.

11. (*a*) In a right-angled triangle the hypotenuse is 12 ft. and the base is 6 ft.; calculate the ratios specified in Ex. 8, Art. 8.

(*b*) What are these ratios when the lengths in (*a*) are taken twice, three times, one-half as great? Compare, if possible, the *angles* in these triangles.

12. (*a*) In a right-angled triangle the base is 8 in. and the perpendicular 12 in.; calculate the ratios specified in Ex. 8, Art. 8.

(*b*) What are these ratios when the lengths in (*a*) are taken one-third, and four times as great? Compare, if possible, the *angles* in these triangles.

13. (*a*) In a right-angled triangle the hypotenuse is 35 yd. and the perpendicular is 15 yd.; calculate the ratios specified in Ex. 8, Art. 8.

(*b*) What are these ratios when the lengths in (*a*) are taken four times, six times, one-fifth as great? Compare, if possible, the *angles* in these triangles.

10. Linear measure. Drawing to scale. Direct measurement by means of drawings. Various systems of linear measurement are described in arithmetic. The system mostly used in English-speaking countries is that in which length is given in miles, yards, feet, or inches. The system which is in common use on the continent of Europe, and which is mainly employed in scientific measurements throughout the world, is the metric system. In this system lengths are given in centimetres, metres, etc., the centimetre being a hundredth part of a metre. A metre is equal to 39.37 . . . inches.*

Drawing to scale. It is often desirable to have a drawing on paper which shall serve to give an accurate idea of the relations of certain lines and positions. Maps and architects' plans are familiar examples of such drawings. In a map an inch may represent 1 mile, 10 miles, 100 miles, 500 miles, and so on, according to *the scale* on which the map is made; in a building plan an inch may represent 10 feet, 12 feet, and so on. The operation of drawing on paper lines that shall be a half, a quarter, a tenth, a thousandth, etc., part of the actual length of given lines, is called *drawing to scale.* In many cases the drawings of objects cannot be made full size; for instance, the map of a town, the floor plan of a church; these are drawn to a *reduced* scale. In other cases the drawings are made larger than the actual objects, for instance, the drawings of the minute things that live in a drop of water, the drawings of the various parts of a flower;

* The metric system has the great advantage of being a decimal system. At the present time committees of scientific societies in England and America are working to have the common system replaced by the metric.

these are drawn to an *enlarged* scale. *When a drawing is made to scale, the scale should always be indicated on it.* This may be done in various ways. Thus a mere statement may be made, *e.g.*,

1 inch to 10 feet;

or, the scale may be indicated by a *fraction* which gives the ratio of any line in the drawing to the actual line represented. The scale can also be *shown graphically* by means of a specially marked line. Both the latter methods are illustrated, for instance, on the map of the Kingdom of Saxony in *The Times Atlas:*

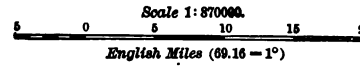

FIG. 1.

The scale should be expressed fractionally, that is, by expressing the ratio of a line in the drawing to the actual line represented. Thus in the first example above the scale is 1:120; in the second the scale is 1:870000.

When a drawing is made to scale, the distance between two objects can be measured directly, by merely measuring the distance between their corresponding points on the drawing. For instance, if 1 inch represents 120 feet, then 2.5 inches represents 300 feet. Another example: On the map of Saxony referred to above, the distance between Leipzig and Dresden is, approximately, $4\frac{1}{2}$ inches, and $4\frac{1}{2}$ inches $\times$ 870000 gives about 62 miles as the distance in an air-line between these cities. This method of finding distance can be used in solving many of the problems in trigonometry.* To find the length of the representative line in the drawing when the scale and the actual length are given, is an exercise in simple proportion; so, also, to find the actual length of a line when the scale and the length of the representative line are known.

* This is one of the methods which will be employed in this book in problems involving distance. Proficiency in drawing will be very helpful to the student.

EXAMPLES.

1. When an inch represents 10 ft., how long must the lines be that will represent 3 in., 6 in., 1 ft., 2 ft., 5 ft., 15 ft., 7.5 ft., 30 ft., 40 ft., 55 ft.? What is the scale?

2. When an inch represents 5 yd., how long must the lines be that will represent 2 yd., 4 yd., 7 yd., 11 yd., 3 yd. 2 ft., 4 yd. 1 ft. 8 in.? What is the scale?

3. When an inch represents 150 ft., what distances are represented by $\frac{1}{8}$ in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., $1\frac{1}{4}$ in., $2\frac{1}{2}$ in., 4.8 in., 5.3 in.? What is the scale?

4. When an inch represents 10 mi., what distances are represented by 3 in., 7 in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., $3\frac{1}{2}$ in.? What lengths on the drawings will represent 7 mi., 18 mi., 25 mi.? What is the scale?

5. What are the scales when 1 in. represents 100 ft., $\frac{1}{4}$ in. represents a mile, $\frac{1}{8}$ in. represents 20 ft., $\frac{1}{16}$ in. represents 15 yd., 1 in. represents 1 mi., 10 mi., 100 mi.?

6. Draw to a scale 1 : 240 (20 ft. to the inch) the circles in Exs. 2, 3, 4, 5, Art. 9.

7. On a map in Baedeker's Guide to Paris the distance between the nearest corners of the Eiffel Tower and Notre Dame Cathedral is $7\frac{3}{4}$ in. What is the distance between those points, the map being drawn to a scale 1 : 20000?

8. Make the comparison of angles asked for in Exs. 8, 9, 10, Art. 8; Exs. 11, 12, 13, Art. 9.

SUGGESTED EXERCISES. Make drawings to scale of the floor plan of a dwelling house, of some other building, of some grounds. Find the distances between various points, such as diagonally opposite corners, by making measurements in the drawing and applying the scale. Compare the results obtained in this way with the results obtained by other methods. Other methods that may be used are: (1) making an off-hand estimate of the distance; (2) actually measuring the distance by "pacing off" or by using a rule or tape line; (3) making a computation. Let the student, from *his own* experience, form a judgment as to which of the four methods referred to is the easiest, and which the more exact. Find the air-line distances between places by measuring the distances between them on maps. Several maps may by used so as to have a variety of scales.

NOTE. The word *scale* also has another meaning in drawing and measurement. Engineers and draughtsmen use various kinds of rules called scales. The faces of these rules contain different numbers of divisions to an inch, one 10 divisions, one 20, one 30, and so on; and generally, one inch on each face is subdivided so that a small fraction of an inch may be set off or read. Some paper scales are on the protractor inserted in this book.

11. Degree measure. The protractor. It has been seen in geometry that: (1) When one line is perpendicular to another line, each of the angles made at their intersection is a right angle; (2) All right angles are equal to one another. In some geometrical propositions angles are compared, and one angle is shown to be greater or less than another. But geometry, with the exception of a few cases, does not show *by exactly how much* the one angle is greater or less than the other. In order to show this, measurement is necessary; and in order to measure, a unit angle of measurement must be chosen. The unit of angular magnitude which is generally used in practical work is the angle that is one-ninetieth part of a right angle. *This unit angle is called a degree.* All degrees are equal to one another, since all right angles are equal to one another. Each degree is subdivided into 60 equal parts called *minutes*, and each minute is subdivided into 60 equal parts called *seconds*. Hence comes the following *table of angular measure:*

60 seconds = 1 minute,
60 minutes = 1 degree,
90 degrees = 1 right angle.

The magnitude of an angle containing 37 degrees and 42 minutes and 35 seconds, say, is written thus: $37° 42' 35''$, read 37 degrees, 42 minutes, 35 seconds. This system of measurement is sometimes called *the rectangular system*, sometimes the *sexagesimal system*. In this chapter only acute angles, that is, angles which contain between 0° and 90°, are considered. Chapter V. considers angles of all magnitudes.

NOTE 1. An angle 1° is subtended by 1 in. at a distance 4 ft. 9.3 in., and by 1 ft. at a distance 57.3 ft. An angle $1'$ is subtended by 1 in. at a distance 286.5 ft., and by 1 ft. at a distance 3437.6 ft., about two-thirds of a mile. An angle $1''$ is subtended by 1 in. at a distance of nearly $3\frac{1}{4}$ mi., by 1 ft. at a distance a little greater than $39\frac{1}{2}$ mi., by a horizontal line 200 ft. long on the other side of the world, nearly 8000 mi. away. These facts can be verified later. See EX. 3, Art. 83.

NOTE 2. Another system of angular measurement was advocated by Briggs and other mathematicians (see Art. 1), and was introduced in France at the time of the Revolution. In this system, which is a decimal one and called *the centesimal system*, a right angle is divided into 100 equal parts called *grades*, each grade into 100 equal parts called *minutes*, and each minute into 100 equal parts called *seconds*. It has not been generally adopted, on account of the immense amount of labour that would be necessary in order to change the mathematical tables computed for the other system.

NOTE 3. The sexagesimal system (from *sexagesimus*, sixtieth) was invented by the Babylonians, who constructed their tables of weights and measures on a scale of 60. Their tables of time (1 day = 24 hr., 1 hr. = 60 min., 1 min. = 60 sec.) and circular measure have come down to the present day. It has been suggested that their adoption of the scale of 60 is due to the fact that they reckoned the year at 360 days. "This led to the division of the circumference of a circle into 360 degrees, each degree representing the daily part of the supposed yearly revolution of the sun around the earth. Probably they knew that the radius could be applied to the circumference as a chord six times, and that each arc thus cut off contained 60 degrees. Thus the division into 60 parts may have suggested itself. . . . Babylonian science has made its impress upon modern civilization. Whenever a surveyor copies the readings from the graduated circle on his theodolite, whenever the modern man notes the time of day, he is, unconsciously perhaps, but unmistakably, doing homage to the ancient astronomers on the banks of the Euphrates." — Cajori, *History of Elementary Mathematics*, pp. 10, 11.

NOTE 4. Another system of angular measure is described in Chapter IX. See Art. 71.

The protractor. The protractor is an instrument used for measuring given angles and laying off required angles on paper. Protractors are of various kinds, of which the semicircular and the full-circled are the most common. The degrees are marked all round the edge. A paper protractor is inserted in this book for use in solving problems.* In order to draw a line that shall make a given angle with a given line at a given point, proceed as follows: Place the centre of the protractor at the given point and bring its diameter into coincidence with the given line, keeping the semicircle on the side on which the required line is to be drawn; prick off the required number of degrees with a sharp pencil or fine needle. The line joining the point thus fixed and the given point, is the line required. In order to measure a given angle with the protractor, place the centre at the vertex of the angle, and place the diameter in coincidence with one of the boundary lines of the angle; the number of degrees in the arc intercepted between the boundary lines of the angle is the measure of the angle.

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* A horn protractor costs about 25 cents, and a small metal one about 50 cents. One who is neat and handy can make a paper protractor.

NOTE. Before proceeding further, the student should be able to draw with ease a right-angled triangle, having been given: (*a*) The hypotenuse and a side; (*b*) the two sides about the right angle; (*c*) the hypotenuse and one of the acute angles; (*d*) one of the sides about the right angle and the opposite angle; (*e*) one of the sides about the right angle and the adjacent angle. It is here taken for granted that these problems have been considered in a course in plane geometry or in a course of geometrical drawing.

EXAMPLES.

N. B. *The student is advised to do Exs. 1–6 carefully, and to preserve the results, for they will soon be required for purposes of illustration.*

1. Draw to scale the triangles considered in Exs. 8, 9, 10, Art. 8, and Exs. 11, 12, 13, Art. 9, and measure the angles.

2. Make drawings, on two different scales, of a right-angled triangle whose base is 20 ft. and adjacent acute angle is 55°. In each drawing measure the remaining parts and thence deduce the unknown parts of the original triangle. In each drawing calculate the ratios specified in Ex. 8, Art. 8.

3. Same as Ex. 2, for a right-angled triangle whose hypotenuse is 30 ft. and angle at base is 25°.

4. Same as Ex. 2, for a right-angled triangle whose base and perpendicular are 30 ft. and 45 ft. respectively.

5. Same as Ex. 2, for a right-angled triangle whose hypotenuse is 60 ft. and base is 45 ft.

6. Same as Ex. 2, for a right-angled triangle in which the base is 50 ft. and the angle opposite to the base is 40°.

7. What angles of a whole number of degrees can *easily* be constructed geometrically without the aid of the protractor? Make the constructions.

12. Trigonometric ratios defined for acute angles. The ratios referred to at the beginning of Art. 8 will now be explained so far as acute angles are concerned. (Before proceeding, the student

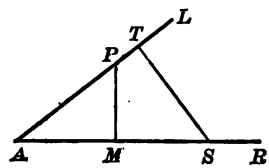

FIG. 2.

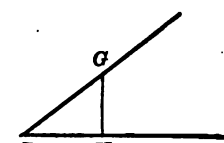

FIG. 3.

should glance over the work on Exs. 8–10, Art. 8; Exs. 11–13, Art. 9; Exs. 1–6, Art. 11.) Let A be any acute angle. In either one of the lines containing the angle take any point P and let fall a perpendicular PM to the other line. The three lines AP, AM, MP, can be taken by twos in three different ways, and hence *six ratios* can be formed with them, namely:

$$\frac{MP}{AP}, \quad \frac{AM}{AP}, \quad \frac{MP}{AM}, \quad \frac{AM}{MP}, \quad \frac{AP}{AM}, \quad \frac{AP}{MP}.$$

It is shown in Art. 13 that each of these ratios has the same value as in Fig. 2, no matter where the point P is taken on either one of the lines bounding an angle which is equal to A. *For the sake of convenience of reference, each one of these six ratios is given a particular name with respect to the angle A.* Thus:

$$\left.\begin{aligned}
&\frac{MP}{AP} \text{ is called the } \textbf{sine} \text{ of the angle } A;\\
&\frac{AM}{AP} \text{ is called the } \textbf{cosine} \text{ of the angle } A;\\
&\frac{MP}{AM} \text{ is called the } \textbf{tangent} \text{ of the angle } A;\\
&\frac{AM}{MP} \text{ is called the } \textbf{cotangent} \text{ of the angle } A;\\
&\frac{AP}{AM} \text{ is called the } \textbf{secant} \text{ of the angle } A;\\
&\frac{AP}{MP} \text{ is called the } \textbf{cosecant} \text{ of the angle } A.
\end{aligned}\right\} \quad \ldots (1)^*$$

These six ratios are known as *the trigonometric ratios* of the angle A. According to the definition of a ratio (Art. 8) they are merely *numbers*. For brevity they are written *sin A*, *cos A*, *tan A*,

* In Chapter V. the trigonometric ratios are defined for angles in general. The definitions give in this article will be found to follow immediately from those given in Art. 40.

cot A, sec A, cosec A (or *csc A*).* Thus *tan A* is read "*tangent A*," and means "the tangent of the angle *A*." The giving of names in (1) may be regarded as defining the trigonometric ratios. Definitions (1) may be expressed as follows:

$$\left.\begin{array}{lll} \dfrac{MP}{AP} = \sin A, & \dfrac{MP}{AM} = \tan A, & \dfrac{AP}{AM} = \sec A, \\ \dfrac{AM}{AP} = \cos A, & \dfrac{AM}{MP} = \cot A, & \dfrac{AP}{MP} = \operatorname{cosec} A, \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

These definitions can be given a slightly different form which is more general, and, accordingly, more useful in applications. In any right-angled triangle *AMP* (Fig. 2), *M* being the right angle, *with reference to the angle A* let *MP* be denoted as the opposite side, and *AM* as the adjacent side. Then these definitions take the form:—

---

* The term *sine* first appeared in the twelfth century in a Latin translation of an Arabian work on astronomy, and was first used in a published work by a German mathematician, *Regiomontanus* (1436–1476). The terms *secant* and *tangent* were introduced by a Dane, *Thomas Finck* (1561–1646), in a work published in 1583. The term *cosecant* seems to have been first used by *Rheticus*, a German mathematician and astronomer (1514–1576), in one of his works which was published in 1596. The names *cosine* and *cotangent* were first employed by *Edmund Gunter* (1581–1626), professor of astronomy at Gresham College, London, who made the first table of logarithms of sines and tangents, published in 1620, and introduced the Gunter's chain now used in land surveying. The abbreviations *sin*, *tan*, *sec*, were first used in 1626 by a Flemish mathematician, Albert Girard (1590–1634), and those of *cos*, *cot*, appear to have been earliest used by an Englishman, William Oughtred (1574–1660), in his *Trigonometry*, published in 1657. These contractions, however, were not generally adopted until after their reintroduction by Leonhard Euler (1707–1783), born in Switzerland of Dutch descent, in a work published in 1748. They were simultaneously introduced in England by Thomas Simpson (1710–1761), professor at Woolwich, in his *Trigonometry*, published in 1748. [See Ball, *A Short History of Mathematics*, pp. 215, 367.] When first used these names referred, not to *certain ratios connected with an angle*, but to *certain lines connected with circular arcs subtended by the angle*. This is explained in Art. 79, which the student can easily read [illegible]me. See Art. 80, Notes 2, 3.

$$\left.\begin{aligned}
\sin A &= \frac{\text{opposite side,}}{\text{hypotenuse,}} \\
\cos A &= \frac{\text{adjacent side,}}{\text{hypotenuse,}} \\
\tan A &= \frac{\text{opposite side,}}{\text{adjacent side,}} \\
\cot A &= \frac{\text{adjacent side,}}{\text{opposite side,}} \\
\sec A &= \frac{\text{hypotenuse,}}{\text{adjacent side,}} \\
\operatorname{cosec} A &= \frac{\text{hypotenuse,}}{\text{opposite side,}}
\end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

[The word *perpendicular* is sometimes used instead of *opposite side*, and *base* instead of *adjacent side*.]

It is necessary that these definitions be thoroughly memorized.

EXAMPLES.

N.B. *The student is requested to preserve the work and results of these Exs. for purposes of future reference.*

1. In AMP (Fig. 2) give the trigonometric ratios of angle APM. Note what ratios of angles A and P are equal.

2. In Figs. 45 *a*, 45 *b*, Art. 46, give the trigonometric ratios of the various acute angles.

3. Find the trigonometric ratios of the acute angles in the triangles in Exs. 8–10, Art. 8; Exs. 11–13, Art. 9; Exs. 2–6, Art. 11.

4. In a triangle PQR right-angled at Q, the hypotenuse PR is 10 in. long, and the side QR is 7. Find the trigonometric ratios of the angles P and R. Note what ratios of P and R are equal.

5. For each of the angles in Ex. 4, and for each of any three of the angles in Ex. 3, calculate the following, and *make a note of the result.* [Let x denote the angle whose ratios are being considered.]

(1) $\sin x \operatorname{cosec} x$ (2) $\cos x \sec x$ (3) $\tan x \cot x$

(4) $\sin^2 x + \cos^2 x$ (5) $\sec^2 x - \tan^2 x$ (6) $\operatorname{cosec}^2 x - \cot^2 x$

(7) $\tan x - \dfrac{\sin x}{\cos x}$ (8) $\cot x - \dfrac{\cos x}{\sin x}$

6. Make the same calculations for angle A in Fig. 2, Art. 12.

13. Definite and invariable connection between (acute) angles and trigonometric ratios. It is important that the following principles be clearly understood:

(1) *To each value of an angle there corresponds but one value of each trigonometric ratio.*

(2) *Two unequal acute angles have different trigonometric ratios.*

(3) *To each value of a trigonometric ratio there corresponds but one value of an acute angle.*

(1) In Fig. 2, Art. 12, from any point S in AR draw ST perpendicular to AL. Let angle B (Fig. 3) be equal to A, and from any point G in one of the lines containing angle B draw GK perpendicular to the other line. Then, by definition (3), Art. 12,

$$\sin A \text{ (in } AMP) = \frac{MP}{AP}, \qquad \sin A \text{ (in } AST) = \frac{ST}{AS},$$

$$\sin B \ (= \sin A) = \frac{KG}{BG}.$$

But the triangles AMP, AST, BKG, are mutually equiangular. Hence the sides about the equal angles are proportional, and

$$\frac{MP}{AP} = \frac{ST}{AS} = \frac{KG}{BG}.$$

Therefore all angles equal to A have the same sine. In like manner, these angles can be shown to have the same tangent, secant, etc.*

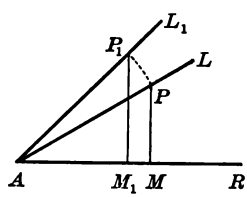

Fig. 4.

(2) Let RAL and RAL_1 be any two unequal acute angles, placed, for convenience, so as to have a common vertex A and a common boundary line AR. From any point P on AL draw PM perpendicular to AR. Take $AP_1 = AP$, and draw P_1M_1 perpendicular to AR. Then

* In Euclid's text on geometry, the properties of similar triangles are considered in Bk. VI. Pupils who study Euclid and have not reached Bk. VI. can be helped to understand these properties by means of a few exercises like those referred to in Ex. 3, Art. 12.

$$\sin RAL = \frac{MP}{AP}, \quad \sin RAL_1 = \frac{M_1P_1}{AP_1}.$$

But $$M_1P_1 > MP, \text{ and } AP_1 = AP;$$

hence $$\sin RAL_1 > \sin RAL.$$

In a similar manner the other ratios can be shown to be respectively unequal.

Ex. In this construction AP_1 is taken equal to AP. Why does this not affect the generality of the proof?

(3) This property follows as a corollary from (1) and (2).

The trigonometric ratios for angles from 0° to 90° are arranged in tables. In some tables the calculations are given to four places of decimals, in others to five, six, or seven places. There are also tables of the logarithms of the ratios (or of the logarithms increased by 10),* which vary in the number of places of decimals to which the calculations are carried out. The student is advised to examine a table of the trigonometric ratios at this time. A good exercise will consist in finding the logarithms of some of the sines, tangents, etc., adding 10 to each logarithm, and comparing the result with that given in the table of Logarithmic sines, tangents, etc. [What are denoted as Natural sines and cosines in the tables, are merely the actual sines and cosines, which have been discussed above; the so-called Logarithmic sines and cosines are the logarithms of the Natural sines and cosines with 10 added.] A book of logarithms and trigonometric ratios is the principal help and tool in solving most of the problems in practical trigonometry; and hence, proficiency in using the tables is absolutely necessary. The larger part of the numerical answers in this book have been obtained with the aid of a five-place table. Those who use six-place or seven-place tables will reach more accurate results.

EXAMPLES.

1. Compare each of the ratios of RAL_1 with the corresponding ratio of RAL.

2. Suppose that the line AR (Fig. 4) revolves about A in a *counter-clockwise direction*, starting from the position AM: show that, as the angle MAL

* These are usually called Logarithmic sines, tangents, etc.

increases, its sine, tangent, and secant increase, and its cosine, cotangent, and cosecant decrease. *Test this conclusion by an inspection of a table of Natural ratios.*

3. Find by tables, sin 17° 40′, sin 43° 25′ 10″, sin 76° 43′, sin 83° 20′ 25″, cos 18° 10′, cos 37° 40′ 20″, cos 61° 37′, cos 72° 40′ 30″, tan 37° 40′ 20″, tan 79° 37′ 30″, cot 42° 30′, cot 72° 25′ 30″. Log sin 37° 20′, Log sin 70° 21′ 30″, Log cos 30° 20′ 20″, Log cos 71° 25′, Log tan 79° 30′ 20″, Log cot 48° 20′ 40″.

4. Find the angles corresponding to the following Natural and Logarithmic ratios:

sine = .15327,	sine = .62175,	sine = .82462,	sine = .84316,
cosine = .85970,	cosine = .61497,	cosine = .84065,	cosine = .60165.
tangent = .42482,	tangent = .60980,	tangent = 1.6820,	tangent = 2.4927,

Log sine = 9.79230, Log sine = 9.94215,

Log cosine = 9.96611, Log cosine = 9.74743,

Log tangent = 9.82120, Log tangent = 10.37340.

14. Practical problems. The problems in this article are intended to help the learner to realize more clearly and strongly the meaning and the usefulness of the ratios which have been defined in Art. 12. The student is earnestly recommended to try to solve the first three problems below *without help* from the book. He will find this to be an advantage, whether he can solve the problems or not. If he *can* solve them, then he will have the pleasurable feeling that he is to some extent independent of the book; and he will thus be encouraged and strengthened for future work. Should he *fail* to solve them, he will have the advantage of a closer acquaintance with the difficulties in the problems, and so will observe more keenly how these difficulties are avoided or overcome. Throughout this course the student will find it to be of immense advantage if he will think and study over the subject-matter indicated in the headings of the articles and make some kind of an attack on the problems *before* appealing to the book for help. If he follows this plan, his progress, in the long run, will be easier and more rapid, and his mental power more greatly improved than if he is content merely to follow after, or be led by, the teacher or author.

EXAMPLES.

1. Construct the acute angle whose cosine is $\frac{2}{3}$. What are its other trigonometric ratios? Find the number of degrees in the angle.

The definition of the cosine of an angle shows that the required angle is equal to an angle in a certain right-angled triangle, namely, the triangle in which "the side adjacent to the angle is to the hypotenuse in the ratio 2 : 3." Thus the lengths of this side and hypotenuse can be taken as 2 and 3, 6 and 9, 200 and 300, and so on. Taking the lengths 2, 3, (these numbers being simpler and, accordingly, more convenient than the others), construct a right-angled triangle AST which has side $AS = 2$, and hypotenuse $AT = 3$. The angle A is the angle required, for $\cos A = \frac{2}{3}$.

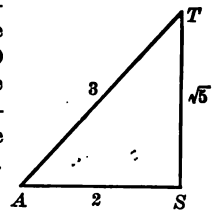

FIG. 5.

Now $$ST = \sqrt{3^2 - 2^2} = \sqrt{5} = 2.2361.$$

Hence, *the other ratios* are

$$\sin A = \frac{\sqrt{5}}{3} = .7454,\ \tan A = \frac{\sqrt{5}}{2} = 1.1180,\ \cot A = \frac{2}{\sqrt{5}} = .8944,$$

$$\sec A = \frac{3}{2} = 1.5000,\ \operatorname{cosec} A = \frac{3}{\sqrt{5}} = 1.3416.$$

The *measure of the angle* can be found in either one of two ways, viz. : (*a*) by measuring the angle with the protractor; (*b*) by finding in the table the angle whose cosine is $\frac{2}{3}$ or .6667. The latter method shows that $A = 48° 11' 22''$. [Compare the result obtained by method (*a*) with the value given by method (*b*).]

2. A right-angled triangle has an angle whose cosine is $\frac{2}{3}$, and the length of the hypotenuse is 50 ft. Find the angles and the lengths of the two sides.

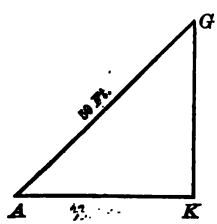

FIG. 6.

By method shown in Ex. 1, construct an angle A whose cosine is $\frac{2}{3}$. On one boundary line of the angle take a length AG to represent 50 ft. Draw GK perpendicular to the other boundary line.

$$\text{Cos } A = \tfrac{2}{3} = .6666 \cdots,$$

$$\therefore\ A = 48° 11' 22'',$$

$$\therefore\ B = 90 - A = 41° 48' 38'',$$

$$\cos A = \frac{AK}{AG} = .6666 \cdots, \qquad \sin A = \frac{\sqrt{5}}{3}, \qquad \text{(Ex. 1)}$$

$$\therefore\ AK = 50 \times .6666 \cdots, \qquad \therefore\ \frac{KG}{AG} = \frac{\sqrt{5}}{3},$$

$$= 33.333 \cdots, \qquad \therefore\ KG = \frac{\sqrt{5}}{3} \times 50 = 37.27 \cdots.$$

The problem may also be solved graphically as follows. Measure angles A, G, with the protractor. Measure AK, KG directly in the figure.

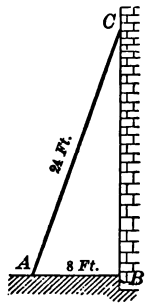

FIG. 7.

3. A ladder 24 ft. long is leaning against the side of a building, and the foot of the ladder is distant 8 ft. from the building in a horizontal direction. What angle does the ladder make with the wall? How far is the end of the ladder from the ground?

Graphical method. Let AC represent the ladder, and BC the wall. Draw AC, AB, to scale, to represent 24 ft. and 8 ft. respectively. Measure angle ACB with the protractor. Measure BC directly in the figure.

Method of computation.

$$BC = \sqrt{AC^2 - AB^2} = \sqrt{576 - 64} = \sqrt{512} = 22.63 \text{ ft.}$$

$$\sin ACB = \frac{AB}{AC} = \frac{8}{24} = .33333,$$

$$\therefore\ ACB = 19° 28' 16''.$$

4. Find tan 40° by construction and measurement. With the protractor lay off an angle SAT equal to 40°. From any point P in AT draw PR perpendicularly to AS. Then measure AR, RP, and substitute the values in the ratio, $\tan 40° = \frac{RP}{AR}$. Compare the result thus obtained with the value given for tan 40° in the tables.*

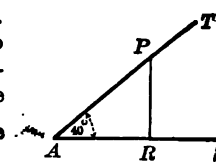

FIG. 8.

5. Construct the angle whose tangent is $\frac{1}{3}$. Find its other ratios. Measure the angle approximately, and compare the result with that given in the tables. Draw a number of right-angled, obtuse-angled, and acute-angled triangles, each of which has an angle equal to this angle.

6. Similarly for the angle whose sine is $\frac{2}{5}$; and for the angle whose cotangent is 3.

7. Similarly for the angle whose secant is $2\frac{1}{2}$; and for the angle whose cosecant is $3\frac{1}{2}$.

8. Find by measurement of lines the approximate values of the trigonometric ratios of 30°, 40°, 45°, 50°, 55°, 60°, 70°; compare the results with the values given in the tables.

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* The values of the ratios are calculated by an algebraic method, and can be found to any degree of accuracy that may be required.

If any of the following constructions asked for is impossible, explain why it is so.

9. Construct the acute angles in the following cases: (*a*) When the sines are $\frac{1}{2}$, 2, $\frac{2}{7}$, $\frac{3}{5}$; (*b*) when the cosines are $\frac{1}{2}$, $\frac{4}{5}$, 3, .3; (*c*) when the tangents are 3, 4, $\frac{2}{3}$, $\frac{1}{4}$; (*d*) when the cotangents are 4, 2, $\frac{2}{3}$, .7; (*e*) when the secants are 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, $4\frac{1}{2}$; (*f*) when the cosecants are 3, 2.5, .4, $\frac{3}{5}$.

10. Find the other trigonometric ratios of the angles in Ex. 9. Find the measures of these angles, (*a*) with the protractor, (*b*) by means of the tables.

11. What are the other trigonometric ratios of the angles: (1) whose sine is $\frac{a}{b}$; (2) whose cosine is $\frac{a}{b}$; (3) whose tangent is $\frac{a}{b}$; (4) whose cotangent is $\frac{a}{b}$; (5) whose secant is $\frac{a}{b}$; (6) whose cosecant is $\frac{a}{b}$?

12. A ladder 32 ft. long is leaning against a house, and reaches to a point 24 ft. from the ground. Find the angle between the ladder and the wall.

13. A man whose eye is 5 ft. 8 in. from the ground is on a level with, and 120 ft. distant from, the foot of a flag pole 45 ft. 8 in. high. What angle does the direction of his gaze, when he is looking at the top of the pole, make with a horizontal line from his eye to the pole?

14. Find the ratios of 45°, 60°, 30°, 0°, 90°, *before* reading the next article.

15. Trigonometric ratios of 45°, 60°, 30°, 0°, 90°. The ratios of certain angles which are often met will now be found.

A. **Ratios of 45°.** Let AMP be an isosceles right-angled triangle, and let each of the sides about the right angle be equal to a.

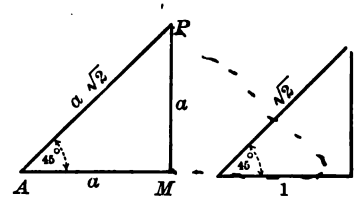

FIG. 9.

FIG. 10.

The angle

$$A = 45°, \text{ and } AP = a\sqrt{2}.$$

$$\therefore \sin 45° = \sin A = \frac{MP}{AP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

By using the same figure it can be shown that

$$\cos 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1, \qquad \cot 45° = 1,$$

$$\sec 45° = \sqrt{2}, \qquad \operatorname{cosec} 45° = \sqrt{2}.$$

The sides of triangle AMP are proportional to 1, 1, $\sqrt{2}$. Hence, in order to produce the ratios of 45° quickly, it is merely necessary to draw Fig. 10; from this figure the ratios of 45° can be read off at once.

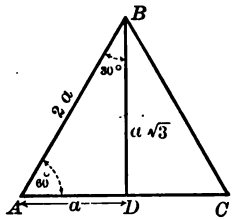

Fig. 11.

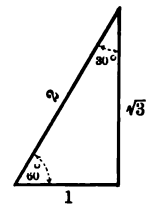

Fig. 12.

B. **Ratios of 30° and 60°.** Let ABC be an equilateral triangle. From any vertex B draw a perpendicular BD to the opposite side AC. Then angle $DAB = 60°$, angle $ABD = 30°$.

If $AB = 2a$, then $AD = a$, and $DB = \sqrt{4a^2 - a^2} = a\sqrt{3}$.

$$\therefore \sin 60° = \sin DAB = \frac{DB}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}.$$

By using the same figure it can be shown that

$$\cos 60° = \tfrac{1}{2}, \qquad \tan 60° = \sqrt{3}, \qquad \cot 60° = \frac{1}{\sqrt{3}},$$

$$\sec 60° = 2, \qquad \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

Also, $$\sin 30° = \sin ABD = \frac{AD}{AB} = \frac{a}{2a} = \tfrac{1}{2}.$$

Similarly,

$$\cos 30° = \frac{\sqrt{3}}{2}, \qquad \tan 30° = \frac{1}{\sqrt{3}}, \qquad \cot 30° = \sqrt{3},$$

$$\sec 30° = \frac{2}{\sqrt{3}}, \qquad \operatorname{cosec} 30° = 2.$$

In ADB the sides opposite to the angles 30°, 60°, 90°, are respectively proportional to 1, $\sqrt{3}$, 2. Hence, in order to produce the ratios of 30°, 60°, at a moment's notice, it is merely necessary to draw Fig. 12, from which these ratios can be immediately read off.

C. **Ratios of 0° and 90°.** *The algebraical note, Art. 76, may be read now.*

Let the hypotenuse in each of the right-angled triangles in Fig. 13 be equal to a.

$$\text{Sin } MAP = \frac{MP}{AP},$$

$$\cos MAP = \frac{AM}{AP},$$

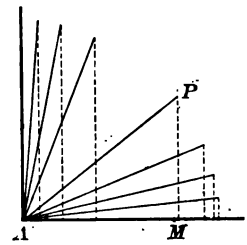

FIG. 13.

It is apparent from this figure that if the angle MAP approaches zero, then the perpendicular MP approaches zero, and the hypotenuse AP approaches to an equality with AM; so that, finally, if $MAP = 0$, then $MP = 0$, and $AP = AM$. Therefore, when $MAP = 0$, it follows that:

$$\sin 0° = \frac{0}{a} = 0, \qquad \tan 0° = \frac{0}{a} = 0, \qquad \sec 0° = \frac{a}{a} = 1,$$

$$\cos 0° = \frac{a}{a} = 1, \qquad \cot 0° = \frac{a}{0} = \infty, \qquad \operatorname{cosec} 0° = \frac{a}{0} = \infty.$$

As MAP approaches 90°, AM approaches zero, and MP approaches to an equality with AP. Therefore, when $MAP = 90°$, it follows that:

$$\sin 90° = \frac{a}{a} = 1, \qquad \tan 90° = \frac{a}{0} = \infty, \qquad \sec 90° = \frac{a}{0} = \infty,$$

$$\cos 90° = \frac{0}{a} = 0, \qquad \cot 90° = \frac{0}{a} = 0, \qquad \operatorname{cosec} 90° = \frac{a}{a} = 1.$$

EXAMPLES.

N. B. Read the first few lines of Art. 17 before attacking the problems.

Find the numerical value of

1. $\sin 60° + 2 \cos 45°$.
2. $\sec^2 30° + \tan^3 45°$.
3. $\sin^3 60° + \cot^3 30°$.
4. $\cos 0° \sin 45° + \sin 90° \sec^2 30°$.
5. $4 \cos^2 30° \sin^2 60° \cos^2 0°$.
6. $3 \tan^3 30° \sec^3 60° \sin^2 90° \tan^2 45°$.
7. $10 \cos^4 45° \sec^6 30°$.
8. $2 \sin^5 30° \tan^3 60° \cos^3 0°$.
9. $x \cot^3 45° \sec^2 60° = 11 \sin^2 90°$; find x.
10. $x(\cos 30° + 2 \sin 90° + 3 \cos 45° - \sin^2 60°) = 2 \sec 0° - 5 \sin 90°$; find x.

16. Relations between the trigonometric ratios of an acute angle and those of its complement. When two angles added together make a right angle, the two angles are said to be *complementary*, and each angle is called *the complement* of the other.

For example, the acute angles in a right-angled triangle are complementary; the complement of A is $90° - A$; the complement of 27° is 63°.

Ex. 1. What are the complements of 10°, 12° 30′, 47°, 56° 27′, 35°?

Ex. 2. What angles are complementary to 23°, 42°, 51°, 78°, 86°?

In Fig. 2, Art. 12, the angle APM is the complement of the angle A. Now,

$$\sin P = \frac{AM}{AP}, \quad \tan P = \frac{AM}{MP}, \quad \sec P = \frac{AP}{MP},$$

$$\cos P = \frac{MP}{AP}, \quad \cot P = \frac{MP}{AM}, \quad \operatorname{cosec} P = \frac{AP}{AM}.$$

Comparison of these ratios with the ratios of A in (2), Art. 12, shows that

$$\cos A = \sin P, \quad \tan A = \cot P, \quad \sec A = \operatorname{cosec} P,$$

$$\sin A = \cos P, \quad \cot A = \tan P, \quad \operatorname{cosec} A = \sec P.$$

These six relations can be expressed briefly:

Each trigonometric ratio of an angle is equal to the corresponding co-ratio of its complement.

Ex. Compare the ratios of 30° and 60°; of 0° and 90°.

17. Exponents in trigonometry.

When a trigonometric ratio has an exponent, a particular way of placing the exponent has been adopted. For example,

$$(\sin x)^2 \text{ is written } \sin^2 x.$$

There is no ambiguity in the second form, and the advantage is apparent. Thus, $\cos^{\frac{1}{2}} x$, $\tan^3 x$, $\sec^{\frac{3}{4}} x$, represent or mean $(\cos x)^{\frac{1}{2}}$, $\sqrt{(\tan x)^3}$, $(\sec x)^{\frac{3}{4}}$. There is one exponent, however, which must not be written with the brackets removed. This exception is the exponent -1. Thus, for example, $(\cos x)^{-1}$, which means $\frac{1}{\cos x}$, must never be written $\cos^{-1} x$. The reason for this is that the symbol $\cos^{-1} x$ is used to represent something else. This symbol denotes *the angle whose cosine is* x, and is read thus, or is read "the

anti-cosine of x," "the inverse cosine of x," "cosine minus one x." The number -1, which appears in $\cos^{-1}x$, is not an exponent at all, but is merely part of a symbol.

Suppose that (*a*) "the sine of the angle A is $\frac{3}{5}$."

The latter idea can also be expressed by saying

(*b*) "A is the angle whose sine is $\frac{3}{5}$";

or, more briefly, by saying,

(*c*) "A is the anti-sine of $\frac{3}{5}$."

The two ways, (*a*), (*c*), of expressing the same idea can be indicated still more briefly by equations, viz.,

$$\sin A = \tfrac{3}{5}, \quad A = \sin^{-1}\tfrac{3}{5}.$$

Thus, $(\sin x)^{-1}$ and $\sin^{-1}x$ mean very different things; for $(\sin x)^{-1}$ is $\frac{1}{\sin x}$, which is *a number*, and $\sin^{-1}x$ is *an angle*.

NOTE. The symbols $\sin^{-1}x$, $\cos^{-1}x$, ..., are considered in Art. 88.

EX. Express $\sin A = \frac{3}{5}$, $\cos x = \frac{3}{4}$, $\tan C = 4$, $\sec A = 9$, $\operatorname{cosec} A = \frac{17}{15}$, in the inverse form.

18. Relations between the trigonometric ratios of an acute angle.

[**N.B.** Some relations between these ratios may have been noticed or discovered by the student in the course of his preceding work. If so, they should now be collected, so that they can be compared with the relations shown in this article.]

Some of the preceding exercises have shown that when *one* trigonometric ratio of an angle is known, the remaining *five* ratios can be easily determined. This at least suggests that the ratios are related to one another. In what follows, A denotes *any* acute angle.

A. Reciprocal relations between the ratios.

Inspection of the definitions (3), Art. 12, shows that:

$$\left.\begin{array}{lll} (a)\ \sin A = \dfrac{1}{\operatorname{cosec} A}, & \operatorname{cosec} A = \dfrac{1}{\sin A}, & \text{or, } \sin A \operatorname{cosec} A = 1; \\ (b)\ \cos A = \dfrac{1}{\sec A}, & \sec A = \dfrac{1}{\cos A}, & \text{or, } \cos A \sec A = 1; \\ (c)\ \tan A = \dfrac{1}{\cot A}, & \cot A = \dfrac{1}{\tan A}, & \text{or, } \tan A \cot A = 1. \end{array}\right\} \quad (1)$$

B. The tangent and cotangent in terms of the sine and cosine.

In the triangle AMP (Fig. 2, Art. 12),

$$\tan A = \frac{MP}{AM} = \frac{\frac{MP}{AP}}{\frac{AM}{AP}} = \frac{\sin A}{\cos A}; \quad \cot A = \frac{AM}{MP} = \frac{\frac{AM}{AP}}{\frac{MP}{AP}} = \frac{\cos A}{\sin A}. \quad (2), (3)$$

C. Relations between the squares of certain ratios.

In the triangle AMP (Fig. 2, Art. 12), indicating by $\overline{MP}^2$ the square of the length of MP,

$$\overline{MP}^2 + \overline{AM}^2 = \overline{AP}^2.$$

On dividing each member of this equation by $\overline{AP}^2$, $\overline{AM}^2$, $\overline{MP}^2$, in turn, there is obtained

$$\left(\frac{MP}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = \left(\frac{AP}{AP}\right)^2,$$

$$\left(\frac{MP}{AM}\right)^2 + \left(\frac{AM}{AM}\right)^2 = \left(\frac{AP}{AM}\right)^2,$$

$$\left(\frac{MP}{MP}\right)^2 + \left(\frac{AM}{MP}\right)^2 = \left(\frac{AP}{MP}\right)^2.$$

In reference to the angle A, these equations can be written:

$$\left.\begin{aligned} \sin^2 A + \cos^2 A &= 1, \\ \tan^2 A + 1 &= \sec^2 A, \\ 1 + \cot^2 A &= \operatorname{cosec}^2 A. \end{aligned}\right\} \quad \ldots \quad (4)$$

NOTE 1. The relations shown above are true, not only for acute angles, but for all angles. This is shown in Art. 44.

NOTE 2. Relations (1) have a practical bearing on the construction and the use of tables. Thus, for example, since $\cos A = \frac{1}{\sec A}$, a table of natural cosines can be transformed into a table of natural secants by merely taking the reciprocals of the cosines. Again, in logarithmic computation, since $\sec A = \frac{1}{\cos A}$, $\log \sec A = -\log \cos A$.

NOTE 3. An equation involving trigonometric ratios is a *trigonometric equation*. Thus, for example, $\tan A = 1$. *One* angle which satisfies this equation is the *acute* angle $A = 45°$. Other solutions can be found after Arts. 84–87 have been taken up.

EXAMPLES.

A few simple exercises are given below, the solution of which brings in the relations shown in this article. These exercises are *algebraic* in character; collections of exercises of this kind are given also in other places in this book. In the following examples, the positive values of the radicals are to be taken. The meaning of the negative values is shown in Art. 44.

1. Given that $\sin A = \frac{1}{2}$, find the other trigonometric ratios of A by means of the relations shown in this article.

$$\operatorname{cosec} A = \frac{1}{\sin A} = 2\,;\ \cos A = \sqrt{1-\sin^2 A} = \frac{\sqrt{3}}{2};\ \sec A = \frac{1}{\cos A} = \frac{2}{\sqrt{3}};$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\sqrt{3}};\quad \cot A = \frac{1}{\tan A} = \sqrt{3}.$$

These results may be verified by the method used in solving Exs. 1, 5–7, Art. 14.

2. Express all the ratios of angle A in terms of $\sin A$.

$$\sin A = \sin A\,;\ \cos A = \sqrt{1-\sin^2 A}\,;\ \tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}};$$

$$\cot A = \frac{1}{\tan A} = \frac{\sqrt{1-\sin^2 A}}{\sin A};\ \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}};\ \operatorname{cosec} A = \frac{1}{\sin A}.$$

3. Prove that $\dfrac{1}{1-\sin A} + \dfrac{1}{1+\sin A} = 2\sec^2 A.$

$$\frac{1}{1-\sin A} + \frac{1}{1+\sin A} = \frac{2}{1-\sin^2 A} = \frac{2}{\cos^2 A} = 2\sec^2 A.$$

4. Prove that $\sec^4 A - 1 = 2\tan^2 A + \tan^4 A.$

$$\sec^4 A - 1 = (\sec^2 A)^2 - 1 = (1+\tan^2 A)^2 - 1 = 2\tan^2 A + \tan^4 A.$$

5. Solve the equation $4\sin\theta - 3\operatorname{cosec}\theta = 0.$

$$4\sin\theta - \frac{3}{\sin\theta} = 0.$$

$$\therefore\ 4\sin^2\theta - 3 = 0,$$

$$\therefore\ \sin^2\theta = \frac{3}{4},\quad \therefore\ \sin\theta = +\frac{\sqrt{3}}{2},\ \text{and}\ \sin\theta = -\frac{\sqrt{3}}{2}.$$

On taking the *plus* sign, *one* solution is the acute angle $\theta = 60°$; other solutions will be found later. For the *minus* sign there is also a set of solutions; these will be found later.

6. Solve $2\sin^2\theta \operatorname{cosec}\theta - 5 + 2\operatorname{cosec}\theta = 0,$

$$\frac{2\sin^2\theta}{\sin\theta} - 5 + \frac{2}{\sin\theta} = 0,$$

$$2\sin^2\theta - 5\sin\theta + 2 = 0,$$

$$(2\sin\theta - 1)(\sin\theta - 2) = 0.$$

$\therefore \sin\theta = \frac{1}{2}$, and $\sin\theta = 2$.

The acute angle whose sine is $\frac{1}{2}$ is 30°; hence $\theta = 30°$ is *one* solution. The sine cannot exceed unity; hence $\sin\theta = 2$ does not afford any solution.

7. Given $\cos A = \frac{3}{4}$, $\sin B = \frac{2}{3}$, $\tan C = 2$, $\cot D = \frac{1}{2}$, $\sec E = 3$, $\operatorname{cosec} F = 2.5$; find the other trigonometric ratios of A, B, C, D, E, F, by the algebraic method. Verify the results by the method used in Art. 14.

8. Find by the algebraic method the ratios required in Exs. 1, 5–7, 10, 11, Art. 14.

9. Express all the trigonometric ratios of an angle A in terms of: (*a*) $\cos A$; (*b*) $\tan A$; (*c*) $\cot A$; (*d*) $\sec A$; (*e*) $\operatorname{cosec} A$. Arrange the results and those of Ex. 2, neatly in tabular form.

Prove the following identities:

10. $(\sec^2 A - 1)\cot^2 A = 1$; $\cos A \tan A = \sin A$; $(1 - \sin^2 A)\sec^2 A = 1$.

11. $\sin^2\theta \sec^2\theta = \sec^2\theta - 1$; $\tan^2\theta - \cot^2\theta = \sec^2\theta - \operatorname{cosec}^2\theta$.

12. $$\frac{1}{\sec^2 A} + \frac{1}{\operatorname{cosec}^2 A} = 1; \qquad \frac{\sin A}{\operatorname{cosec} A} + \frac{\cos A}{\sec A} = 1;$$

$$(\tan\theta + \sec\theta)^2 = \frac{1 + \sin\theta}{1 - \sin\theta}$$

13. $$\sec^2 A + \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2;$$

$$\frac{1 + \tan^2 A}{1 + \cot^2 A} = \frac{\sin^2 A}{\cos^2 A}; \qquad \frac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A.$$

14. $$\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A;$$

$$\frac{1}{\sec A - \tan A} = \sec A + \tan A; \quad \sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$$

Solve the following equations:

15. $2 \sin \theta = 2 - \cos \theta$.

16. $\tan \theta + \cot \theta = 2$.

17. $\tan \theta + 3 \cot \theta = 4$.

18. $6 \sec^2 \theta - 13 \sec \theta + 5 = 0$.

19. $8 \sin^2 \theta - 10 \sin \theta + 3 = 0$.

20. $\sin \theta + 2 \cos \theta = 2.2$.

19. Summary. In this chapter important additions have been made to the knowledge concerning angles that one gained in geometry. A process of measuring angles has been introduced. The close connection between angles and the ratios of lines has been emphasized. It has been shown that each (acute) angle has, associated with it, a definite set of six numbers, called trigonometric ratios; and it has been seen that the sets of numbers are different for different angles. It has also been shown that the seven quantities (namely, the angle and the six numbers) are so related, that, *if one of the seven be given, then the remaining six can be determined.*

A few applications to the measurement of lines and angles have been made in some of the preceding articles. The next two chapters are taken up with a *formal* treatment of such applications. It should be stated, however, that any one who understands the contents of this chapter is *in possession of all the principles* which will be used in the next two chapters, and can proceed directly to the solution of the problems given there. The student is recommended to attack some of the exercises in Chapters III., IV., *before* reading the explanations given in the text. Attention may again be given to the first part of Art. 14.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter II. will be found at pages* 182, 183.

CHAPTER III.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

Before the solution of right-angled triangles is entered upon, a few remarks will be made on the solution of triangles in general. Some of the ideas expressed in Arts. 20–24 are applicable to practical problems throughout the book.

20. Solution of a triangle. Every triangle has three sides and three angles. These six quantities are called the *parts* or *elements* of a triangle. Sometimes one or several of the parts of a triangle are known; for instance, the three sides, two angles and a side, two sides, one side, three angles, and so on. In such cases the questions arise: Can the remaining parts be found or determined? and, if so, by what method shall this be done? The process of deducing the unknown parts of a triangle from the known, is called *solving the triangle*, or, *the solution of the triangle*. This Chapter and Chapter VII. are concerned with showing, in detail, methods of solving triangles. There are two methods which can be used to find (only approximately, in general) the unknown parts of a triangle when some of its parts are given. These methods are:

(*a*) **The graphical method;**

(*b*) **The method of computation.**

21. The graphical method. This method consists *in drawing* a triangle which has angles equal to the given angles, and sides proportional to, and thus representing the given sides, *and then measuring the remaining parts directly from the drawing.*

For example, a triangle has two sides whose lengths are 10 ft., 5 ft., and the included angle is 28° 30′; the third side and the other angles are required.

The graphical solution is as follows: Construct a triangle QPR having two sides, PQ, PR, representing 10 ft., 5 ft., respectively, on some con-

venient scale, and with their included angle, QPR, equal to 28° 30′, as shown in Fig. 14. Measure the angles PRQ, PQR with the protractor; measure the side RQ and, by reference to the scale, find the length represented by RQ. [The results thus obtained may be compared with those obtainable by the method of computation explained in Arts. 54, 57. The latter results are $R = 128° 26' 46''$, $Q = 23° 3' 14''$, $RQ = 6.092$.]

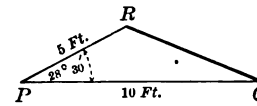

FIG. 14.

The conditions necessary and sufficient for constructing a triangle, and the methods of drawing triangles that satisfy given conditions, are shown in plane geometry and in geometrical drawing. It is obvious that *the graphical method can be employed only when the values of the parts given are consistent with one another, and when the parts given are sufficient in number to determine a definite triangle.* For instance, suppose that one is asked to find the remaining parts of a triangle one of whose sides is 10 inches long. In this case as many unequal triangles as one please, can be constructed, all of which will satisfy the given condition. Again, a given side and a given angle are insufficient *data* on which to proceed to find the remaining parts of a triangle, for there is an infinite number of unequal triangles which can have parts equal to the given parts. So also the method fails if three angles be given; for an infinite number of unequal triangles can be drawn whose angles are equal to the given angles. Again, let it be required to find the angles of a triangle whose sides are 10 feet, 40 feet, 60 feet. Such a triangle is impossible, since the length of one side (60 feet) is greater than the sum of the lengths of the other two. One more instance: let two given angles be 85° and 105° and the included side be 40 inches; this triangle is impossible, since the sum of the two given angles is greater than two right angles.

22. The method of computation. This method is applicable in precisely the same cases in which the preceding method can be employed; namely, in the cases in which the parts given are consistent with one another, and afford conditions sufficient to enable one to construct a definite triangle. This will be fully apparent later, when the various cases will be treated in detail. One of

the principal purposes of this book is to show the different methods of computation applicable to various sets of given conditions. *One of the principal objects* of a student who is taking a first course in trigonometry *should be to acquire facility and, above all, accuracy in using these methods of computation.*

23. Comparison between the graphical method and the method of computation. The experience gained in some of the exercises in the preceding chapter has probably shown the student that he can attain much greater accuracy by using the method of computation than by using the graphical method. The accuracy of the results obtained by the latter method depends upon the carefulness and skill with which the figures are drawn and measured; in the other method, accuracy depends upon the care and patience employed in performing arithmetical work. While the results attainable by the graphical method, even in the case of skilled persons with excellent drawing instruments, are not as accurate as the results attainable by the other method, yet they are often accurate enough for practical purposes. When the computations required are long and complicated, the graphical method is much the more rapid of the two.

There are several reasons why it is advisable for the learner to use the graphical method, as well as the method of computation, in solving problems in this course. The first reason is that the former method *will serve as a check* upon the latter. With ordinary care the graphical method very quickly gives a fair approximation to the result. This result will sometimes show that there is an error in the result obtained by computation. A little error in arithmetic may yield a quantity which is ten times too great or too small; but this can be detected at once if the other method has also been used.* A second reason for using the graphical method is that this method incidentally *provides training in neat, careful, and accurate drawing;* this training will not only be a benefit in itself, but will be of very great advantage in other studies, and especially in the applied sciences. A third

* The results can also be tested by methods of computation, which will be shown in due course

reason is that the pupil *will gain some knowledge and experience of a method that is used in other subjects*, for instance, in physics and in mechanics, and that is extensively employed by engineers in solving problems in which the computations required by the other method may be overwhelmingly cumbrous.

24. General directions for solving problems. A third method of approximating to the magnitude of lines and angles may be mentioned here, for it has often to be employed in practical life. In this method the student may suppose that he possesses neither measuring instruments, drawing materials, nor mathematical tables, and thereupon he *may give an off-hand estimate* concerning the magnitudes required. This method also serves as a check, by showing when great arithmetical blunders are committed. The pupil is advised to use all three methods in working each practical problem in this course, and to do so in the following order:

(1) *Make an off-hand estimate as to what the magnitude required may be, and write this estimate down;*

(2) *Solve the problem by the graphical method;*

(3) *Solve the problem by the slower but more accurate and reliable method of computation.* There may be some interest found in comparing the results obtained by these three methods. The exercise in judging linear and angular magnitudes afforded by the first method, the practice in neat and careful drawing necessary in the second, and the training in accurate computation given by the third, will each afford some benefit to the learner.

25. Solution of right-angled triangles. Let ABC be a right-angled triangle, C being the right angle. In what follows, a, b, c, denote the lengths of the sides opposite to the angles A, B, C, respectively. The sides and angles of ABC are connected by the following relations:

$$\left.\begin{array}{l}(1)\ A+B=90°;\\ (2)\ c^2=a^2+b^2.\end{array}\right\}\ \text{(Geometry)}$$

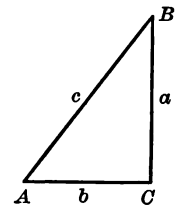

FIG. 15.

$$\left.\begin{array}{ll} (3),\ (4) & \sin A = \frac{a}{c} = \cos B; \\ (5),\ (6) & \cos A = \frac{b}{c} = \sin B; \\ (7),\ (8) & \tan A = \frac{a}{b} = \cot B; \\ (9),\ (10) & \cot A = \frac{b}{a} = \tan B. \end{array}\right\} \quad \therefore \text{ (Definitions (3), Art. 12)}$$

NOTE. To these may be added: $\sec A = \frac{c}{b} = \operatorname{cosec} B$, $\operatorname{cosec} A = \frac{c}{a} = \sec B$. These relations, however, are not needed, and are rarely used in solving triangles, for they are equivalent to (3)–(6), and few tables give secants and cosecants. See Art. 18, Note 2.

Equation (1) shows that no other element of the triangle can be derived from the two acute angles only. Each of the remaining equations, (2)–(10), involves *three* elements of the triangle, and at least two of these elements are sides. Hence, in order that a right-angled triangle be solvable, two elements must be known in addition to the right angle, and one of these must be a side. If any two of the elements involved in equations (2)–(10) are known, then a third element of the triangle can be found therefrom. Hence the following general rule can be used in solving right-angled triangles:

When in addition to the right angle, any two sides, or one of the acute angles and any one of the sides, of a right-angled triangle are known, and another element is required, write the equation involving the required element and two of the known elements, and solve the equation for the required element.

For example, suppose that a, c are known, and that A, B, b are required. In this case,

$$\sin A = \frac{a}{c}, \quad B = 90 - A, \quad b = \sqrt{c^2 - a^2}.$$

The quantities A, B, b can be found from these equations. If an error has been made in finding A, then B will also be wrong. Hence *it is advisable to check the values found* by seeing whether they satisfy relations differing from those already employed.

For example, *check formulas* which may be used in this case are:

$$\frac{b}{a} = \tan B, \quad \frac{b}{c} = \cos A.$$

26. Checks upon the accuracy of the computation. As already pointed out, large errors can be detected by means of the off-hand estimate and by the use of the graphical method. The calculated results in any example can be *checked* or *tested* by employing relations which have not been used in computing the results, and examining whether the newly found values satisfy these relations. An instance has been given in the preceding article. The student is advised not to look up the answers until after he has tested his results in this way. Verification by means of check formulas is necessary in cases in which the answers are not given. The testing of the results also affords practice in the use of formulas and in computation. When a check formula is satisfied it is highly probable, but not absolutely certain, that the calculated results are correct.

27. Cases in the solution of right-angled triangles. All the possible sets of two elements that can be made from the three sides and the two acute angles of a right-angled triangle are the following:

(1) The two sides about the right angle.

(2) The hypotenuse and one of the sides about the right angle.

(3) The hypotenuse and an acute angle.

(4) One of the sides about the right angle, and an acute angle.

(5) The two acute angles. (This case has already been referred to.)

Some examples of these cases are solved. The *general method of procedure*, after making an off-hand estimate and finding an approximate solution by the graphical method, is as follows:

First: *Write all the relations (or formulas) which are to be used in solving the problem.*

Second: *Write the check formulas.*

Third: *In making the computations arrange the work as neatly as possible.*

This last is important, because, by attention to this rule, the work is presented clearly, and mistakes are less likely to occur. The computations may be made either with or without the help of logarithms. The calculations can generally be made more easily and quickly by using logarithms.

Note 1. Relations (3), (4), Art. 25, may be written: $a = c \sin A$, $a = c \cos B$. These relations may be thus expressed:

A side of a right-angled triangle is equal to the product of the hypotenuse and the cosine of the angle adjacent to the side.

A side of a right-angled triangle is equal to the product of the hypotenuse and the sine of the angle opposite to the side.

Note 2. Relations (7), (8), Art. 25, may be written: $a = b \tan A$, $a = b \cot B$. These relations may be thus expressed:

A side of a right-angled triangle is equal to the product of the other side and the tangent of the angle opposite to the first side.

A side of a right-angled triangle is equal to the product of the other side and the cotangent of the angle adjacent to the first side.

EXAMPLES.

1. In the triangle ABC, right-angled at C, $a = 42$ ft., $b = 56$ ft. Find the hypotenuse and the acute angles.

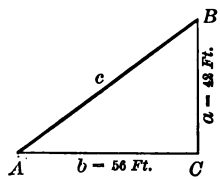

Fig. 16.

I. *Computation without logarithms.* [Four-place tables.]

$\tan A = \frac{a}{b} = \frac{42}{56} = .7500.$ $\therefore A = 36° 52'.2.$

$B = 90 - A.$ $\therefore B = 53° 7'.8.$

$c = \sqrt{a^2 + b^2} = \sqrt{1764 + 3136}.$ $\therefore c = 70$ ft.

Check: $a = c \cos B = 70 \times \cos 53° 7'.8 = 70 \times .6000 = 42$ ft.

II. *Computation with logarithms.*

Given: $a = 42$ ft., $b = 56$ ft.

To find:* $A =$, $B =$, $c =$

Formulas: $\tan A = \frac{a}{b}$. (1)

$B = 90° - A.$ (2)

$c = \frac{a}{\sin A}$. (3)

Checks: $\tan B = \frac{b}{a}$.

$a^2 = c^2 - b^2$

$= (c + b)(c - b)$.

* This is to be filled after the values of the unknown quantities have been found. It is advisable to indicate the given parts and the unknown parts clearly.

Logarithmic formulas: $\log \tan A = \log a - \log b$.

[See Note 6], $\log c = \log a - \log \sin A$.

$\log a = 1.62325$
$\log b = 1.74819$
$\therefore \log \tan A = 9.87506 - 10$
$\therefore A = 36° 52' 12''$
$\therefore B = 53° 7' 48''$

$\log a = 1.62325$
$\log \sin A = 9.77815 - 10$
$\therefore \log c = 1.84510$
$\therefore c = 70$

The work can be more compactly arranged, as follows:

$\log a = 1.62325$
$\log b = 1.74819$
$\therefore \log \tan A = 9.87506 - 10$
$\therefore A = 36° 52' 12''$
$\therefore B = 53° 7' 48''$
$\log \sin A = 9.77815 - 10$
$\therefore \log c = 1.84510$
$\therefore c = 70$

Checks:

$\log \tan B = 10.12494 - 10$
$\therefore B = 53° 7' 48''$
$c + b = 126$
$c - b = 14$
$\log (c + b) = 2.10037$
$\log (c - b) = 1.14613$
$\therefore \log a^2 = 3.24650$
$\therefore \log a = 1.62325$

NOTE 1. The latter form is preferable when all the parts of a triangle are required.

NOTE 2. If there is difficulty in calculating *log a − log sin A* in the second form, write log sin A on the edge of a piece of paper and place it immediately beneath log a.

NOTE 3. The formula $\tan B = \frac{b}{a}$ can be used instead of (2). A check then is $A + B = 90°$. Instead of (3), one of the following formulas can be used, viz.

$$c = \frac{b}{\sin B}, \quad c = \frac{a}{\cos B}, \quad c = \frac{b}{\cos A}.$$

There is often a choice of formulas that can be used in a solution.

NOTE 4. *In every example it is advisable to make a complete* ***skeleton scheme*** *of the solution, before using the tables and proceeding with the actual computation.* In the last exercise, for instance, such a skeleton scheme can be seen on erasing all the numerical quantities in the equations that follow the logarithmic formulas.

NOTE 5. Time will be saved if all the logarithms that can be found at one place in the tables, be written at one time. Thus, for example, in the preceding exercise find log sin A immediately after A has been found.

NOTE 6. The logarithmic formulas can be written on a glance at the formulas such as (1), (2), (3). The writing of the logarithmic formulas may be dispensed with when the student has become familiar with calculation by logarithms. A glance at the original formulas will show how the logarithms are to be combined in the computation.

2. In a triangle ABC right angled at C, $c = 60$ ft., $b = 50$ ft.; find side a and the acute angles.

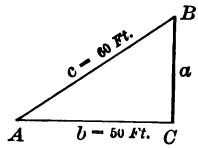

FIG. 17.

I. *Computation without logarithms.*

$$\cos A = \frac{b}{c} = \frac{50}{60} = .8333. \quad \therefore A = 33° 33'.75.$$

$$B = 90° - A. \quad \therefore B = 56° 26'.25.$$

$$a = c \sin A = 60 \times .5528 = 33.17 \text{ ft.}$$

Check: $a = b \tan A = 50 \times .6635 = 33.17.$

II. *Computation with logarithms.*

Given: $c = 60$ ft., $b = 50$ ft.

To find: $A =$, $B =$, $a =$.

Formulas: $\cos A = \frac{b}{c}$. $B = 90° - A$. $a = c \sin A$.

Checks: $a^2 = c^2 - b^2 = (c+b)(c-b)$. $a = b \tan A$.

Logarithmic formulas: $\log \cos A = \log b - \log c$.

(If necessary.) $\log a = \log c + \log \sin A$.

$\log b = 1.69897$ (1)
$\log c = 1.77815$ (2)
$\therefore \log \cos A = 9.92082 - 10$ (3)
$= (1) - (2)$
$\therefore A = 33° 33' 27''$
$\therefore B = 56° 26' 33''$
$\log \sin A = 9.74255 - 10$ (4)
$\therefore \log a = 1.52070$ (5)
$= (2) + (4)$
$\therefore a = 33.16$

$\log \tan A = 9.82173 - 10$ (6)
$\therefore \log a = 1.52070$ (7)
$= (1) + (6)$
$c + b = 110$
$c - b = 10$
$\log (c + b) = 2.04139$
$\log (c - b) = 1$
$\therefore \log a^2 = 3.04139$
$\therefore \log a = 1.52070$

NOTE. There is a slight difference between the results obtained by the two methods. This is due to the fact that the calculations have been made with a four-place table in one case, and with a five-place table in the other. A four-place table will give an angle correctly to within *one minute;* a five-place table will give it correctly to within *six seconds*, and sometimes, to within a second.

EX. Make the computation I. with a five-place table.

3. In a triangle right angled at C, the hypotenuse is 250 ft., and angle A is 67° 30′. Solve the triangle.

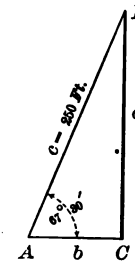

FIG. 18.

I. *Computation without logarithms.*

$B = 90° - A = 90° - 67° 30′ = 22° 30′.$

$a = c \sin A = 250 \times \sin 67° 30′ = 250 \times .9239 = 230.98.$

$b = c \cos A = 250 \times \cos 67° 30′ = 250 \times .3827 = 95.68.$

Checks: $a^2 = c^2 - b^2$, or $a = b \tan A$.

II. *Computation with logarithms.*

Given: $c = 250$ ft., $A = 67° 30′$.

To find: $B =$, $a =$, $b =$

Formulas: $B = 90° - A.$ $a = c \sin A.$ $b = c \cos A.$

Checks: $a^2 = c^2 - b^2$ $= (c + b)(c - b).$

Logarithmic formulas: $\log a = \log c + \log \sin A.$ $\log b = \log c + \log \cos A.$

$\therefore B = 22° 30′$
$\log c = 2.39794$
$\log \sin A = 9.96562 - 10$
$\log \cos A = 9.58284 - 10$
$\therefore \log a = 2.36356$
$\therefore \log b = 1.98078$
$\therefore a = 230.97$
$\therefore b = 95.67$

$c + b = 345.67$
$c - b = 154.33$
$\log (c + b) = 2.53866$
$\log (c - b) = 2.18845$
$\therefore \log a^2 = 4.72711$
$\therefore \log a = 2.36356$

4. In a triangle ABC right angled at C, $b = 300$ ft. and $A = 37° 20′$. Solve the triangle.

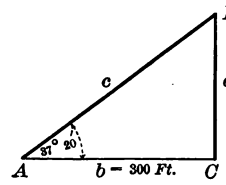

FIG. 19.

I. *Computation without logarithms.*

$B = 90 - A = 90° - 37° 30′ = 52° 40′.$

$c = \frac{b}{\cos A} = \frac{300}{.7951} = 377.3.$

$a = b \tan A = 300 \times .7627 = 228.8.$

Checks: $a^2 = c^2 - b^2$, $a = c \sin A$.

II. *Computation with logarithms.*

Given: $A = 37° 20′$, $b = 300$ ft.

To find: $B =$, $c =$, $a =$

Formulas: $B = 90° - A.$ $c = \frac{b}{\cos A}.$ $a = b \tan A.$

Checks: $a^2 = c^2 - b^2$ $= (c + b)(c - b).$

$\therefore B = 52° 40'$
$\log b = 2.47712$
$\log \cos A = 9.90043 - 10$
$\log \tan A = 9.88236 - 10$
$\therefore \log c = 2.57669$
$\therefore \log a = 2.35948$
$\therefore c = 377.3$
$\therefore a = 228.8$

$c + b = 677.3$
$c - b = 77.3$
$\log (c + b) = 2.83078$
$\log (c - b) = 1.88818$
$\therefore \log a^2 = 4.71896$
$\therefore \log a = 2.35948$

N.B. *Check all results in the following examples.* The given elements belong to a triangle ABC which is right angled at C.

From the given elements solve the following triangles:

5. $c = 18.7$, $a = 16.98$.
6. $a = 194.5$, $b = 233.5$.
7. $c = 2934$, $A = 31° 14' 12''$.
8. $a = 36.5$, $B = 68° 52'$.
9. $a = 58.5$, $b = 100.5$.
10. $c = 45.96$, $a = 1.095$.
11. $c = 324$, $A = 48° 17'$.
12. $b = 250$, $A = 51° 19'$.
13. $c = 1716$, $A = 37° 20' 30''$.
14. $a = 2314$, $b = 1768$.
15. $b = 3741$, $A = 27° 45' 20''$.
16. $c = 50.13$, $a = 24.62$.

Solve Exs. 17–24 by two methods, viz.: (1) with logarithms; (2) without logarithms.

17. $a = 40$, $B = 62° 40'$.
18. $c = 9$, $a = 5$.
19. $a = 4.5$, $b = 7.5$.
20. $c = 15$, $A = 39° 40'$.
21. $c = 12$, $B = 71° 20'$.
22. $c = 12$, $a = 8$.
23. $b = 15$, $B = 42° 30'$.
24. $a = 8$, $b = 12$.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter III. will be found at page* 183.

CHAPTER IV.

APPLICATIONS INVOLVING THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

Some practical applications of trigonometry will now be given. It is not necessary that all the problems be solved, or all the articles be considered, before Chapter V. is taken up.

28. Projection of a straight line upon another straight line. If from a point P a perpendicular PO be drawn to the straight line ST, then O is called *the projection of the point P upon the line ST.* If perpendiculars be drawn from two points A, B, to

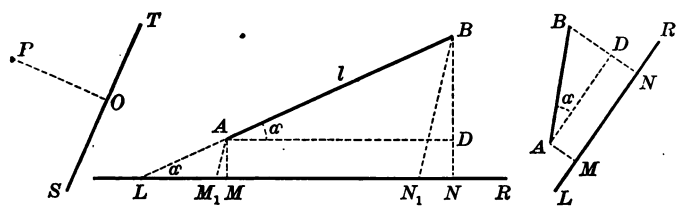

FIG. 20.

a line LR, and intersect LR in M, N, respectively, then MN is called *the projection of AB upon LR.*

Let l be the length of AB, and let α be the angle at which the two lines AB, LR are inclined to each other. Through A draw AD parallel to LR. Then

$$\textbf{Projection} = MN = AD = AB \cos DAB = l \cos \alpha.$$

That is, *the projection of one straight line upon another straight line is equal to the product of the length of the first line and the cosine of the angle of inclination of the two lines.*

NOTE. The projection discussed here, is *orthogonal* (*i.e.* perpendicular) *projection.* If a pair of parallel lines AM_1, BN_1, *not perpendicular* to LR, be drawn through A, B, then M_1N_1 is an *oblique projection* of AB on LR.

EXAMPLES.

In working these examples use logarithms or not, as appears most convenient. Check the results.

1. A ladder 28 ft. long is leaning against the side of a house, and makes an angle 27° with the wall. Find its projections upon the wall and upon the ground.

2. What is the projection of a line 87 in. long upon a line inclined to it at an angle 47° 30′?

3. What are the projections: (*a*) of a line 10 in. long upon a line inclined 22° 30′ to it? (*b*) of a line 27 ft. 6 in. long upon a line inclined 37° to it? (*c*) of a line 43 ft. 7 in. long upon a line inclined 67° 20′ to it? (*d*) of a line 34 ft. 4 in. long upon a line inclined 55° 47′ to it?

29. Measurement of heights and distances. There are various instruments used for measuring angles. The *sextant* can be used for measuring the angle between the two lines drawn from the observer's eye to each of two distant objects. Horizontal and vertical angles can be measured with a *theodolite* or *engineer's transit*. When great accuracy is not required, vertical angles can be measured by means of a *quadrant*.

When an object is above the observer's eye, the angle between the line from the eye to the object, and the horizontal line through the eye and in the same vertical plane as the first line, is called **the angle of elevation** of the object, or simply *the elevation* of the object. When the object is below the observer's eye, this angle is called **the angle of depression** of the object, or simply *the depression* of the object.

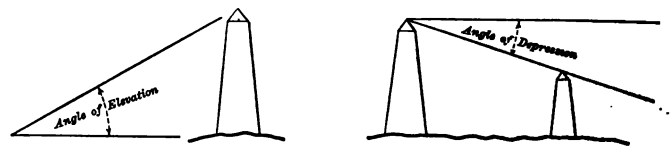

FIG. 21.

NOTE. In ordinary work engineers get angular measurements exact to within one minute, and in the best ordinary work to half a minute. In very particular work, like geodetic survey, they can get measurements exact to five seconds. For ordinary engineering work five-place tables are generally used; four-place tables are used in some kinds of work. See Art. 11, Note 1, Art. 27, Ex. 2, Note.

30. Problems requiring a knowledge of the points of the Mariner's Compass. The circle in the Mariner's Compass is divided into 32 equal parts, each part being thus equal to $360° \div 32$, *i.e.* $11\frac{1}{4}°$. The points of division are named as indicated on the figure.

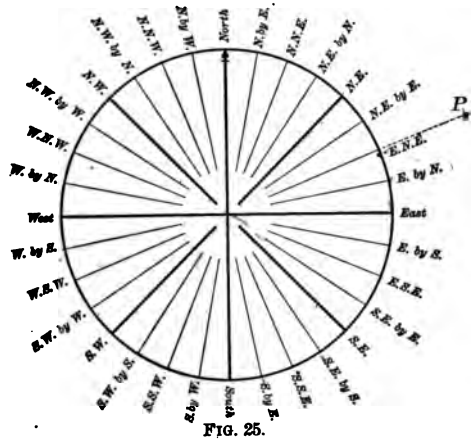

FIG. 25.

It will be observed that the points are named with reference to the points North, South, East, and West, which are called the *cardinal points*. Direction is indicated in a variety of ways. For instance, suppose C were the centre of the circle; then the point P in the figure is said to *bear* E.N.E. from C, or, from C the *bearing* of P is E.N.E. Similarly, C bears W.S.W. from P, or, the bearing of C from P is W.S.W. The point E.N.E. is 2 points North of East, and 6 points East of North. Accordingly, the phrases E. $22\frac{1}{2}°$ N., N. $67\frac{1}{2}°$ E., are sometimes used instead of E.N.E.

EXAMPLES.

1. Two ships leave the same dock at 8 A.M. in directions S.W. by S., and S.E. by E. at rates of 8 and $9\frac{1}{2}$ mi. an hour respectively. Find their distance apart, and the bearing of one from the other at 10 A.M. and at noon.

2. From a lighthouse L two ships A and B are observed in a direction N.E. and N. 20° W. respectively. At the same time A bears S.E. from B. If LA is 6 mi., what is LB?

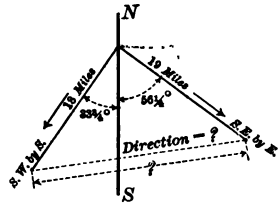

FIG. 26.

31. Mensuration. Let ABC be any triangle, and let the lengths of the sides opposite the angle A, B, C be denoted by a, b, c, respectively. From any vertex C draw CD at right angles to the opposite

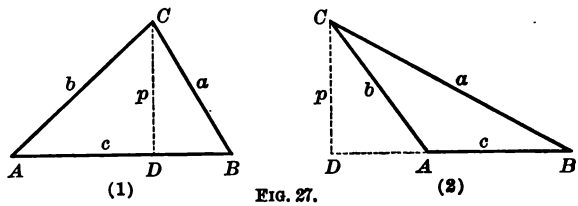

FIG. 27.

side AB. It has been shown in arithmetic and geometry, that the area of a triangle is equal to one-half the product of the lengths of any side and the perpendicular drawn to it from the opposite vertex. [In (1) A is acute, in (2) A is obtuse.]

$$\text{area } ABC \text{ (Fig. 1)} = \tfrac{1}{2} AB \cdot DC;$$
$$= \tfrac{1}{2} AB \cdot AC \sin A;$$
$$= \tfrac{1}{2} bc \sin A.$$

$$\text{area } ABC \text{ (Fig. 2)} = \tfrac{1}{2} AB \cdot DC;$$
$$= \tfrac{1}{2} AB \cdot AC \sin CAD;$$
$$= \tfrac{1}{2} bc \sin (180 - A).$$

It will be seen in Art. 45, that $\sin (180 - A) = \sin A$. Hence, *the area of a triangle is equal to one-half the product of any two sides and the sine of their contained angle.*

EXAMPLES.

1. Find the area of the triangle in which two sides are 31 ft. and 23 ft. and their contained angle 67° 30′.

$$\text{area} = \frac{31 \times 23}{2} \times \sin 67° 30' = \frac{31 \times 23 \times .92388}{2} = 329.37 \text{ sq. ft.}$$

2. Find area of triangle having sides 125 ft., 80 ft., contained angle 28° 35′.

3. Find area of triangle having sides 125 ft., 80 ft., contained angle 151° 25′. [Draw figures carefully for Exs. 2, 3.]

4. Find area of parallelogram two of whose adjacent sides are 243, 315 yd., and their included angle 35° 40′.

5. Find area of parallelogram two of whose adjacent sides are 14, 15 ft., and included angle 75°.

6. Find area of triangle having sides 40 ft., 45 ft., with an included angle 28° 57′ 18″.

7. Write two other formulas for area ABC, similar to that derived above. Also, derive them.

32. Solution of isosceles triangles. In an isosceles triangle, the perpendicular let fall from the vertex to the base bisects the base and bisects the vertical angle. An isosceles triangle can often be solved on dividing it into two equal right-angled triangles.

EXAMPLES.

1. The base of an isosceles triangle is 24 in. long, and the vertical angle is 48° ; find the other angles and sides, the perpendicular from the vertex and the area. Only the steps in the solution will be indicated. Let ABC be an isosceles triangle having base $AB = 24$ in., angle $C = 48°$. Draw CD at right angles to base ; then CD bisects the angle ACB and base AB. Hence, in the right-angled triangle ADC, $AD = \frac{1}{2} AB = 12$, $ACD = \frac{1}{2} ACB = 24°$. Hence, angle A, sides AC, DC, and the area, can be found.

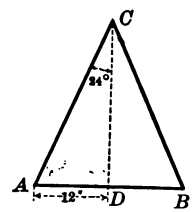

FIG. 28.

2. In an isosceles triangle each of the equal sides is 363 ft., and each of the equal angles is 75°. Find the base, perpendicular on base, and the area.

3. In an isosceles triangle each of the equal sides is 241 ft., and their included angle is 96°. Find the base, angles at the base, height, and area.

4. In an isosceles triangle the base is 65 ft., and each of the other sides is 90 ft. Find the angles, height, and area.

5. In an isosceles triangle the base is 40 ft., height is 30 ft. Find sides, angles, area.

6. In an isosceles triangle the height is 60 ft., one of equal sides is 80 ft. Find base, angles, area.

7. In an isosceles triangle the height is 40 ft., each of equal angles is 63°. Find sides and area.

8. In an isosceles triangle the height is 63 ft., vertical angle is 75°. Find sides and area.

33. Related regular polygons and circles. The knowledge of trigonometry thus far attained, is of service in solving many

problems in which circles and regular polygons are concerned. Some of these problems are:

(*a*) Given the length of the side of a regular polygon of a given number of sides, to find its area; also, to find the radii of the inscribed and circumscribing circles of the polygon;

(*b*) To find the lengths of the sides of regular polygons of a given number of sides which are inscribed in, and circumscribed about, a circle of given radius.

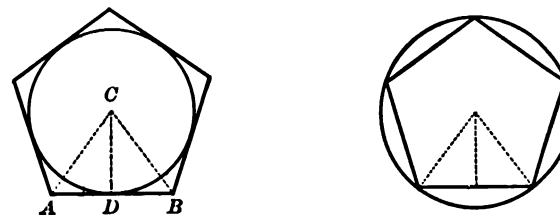

Fig. 29.

For example, let AB (Fig. 29) be a side, equal to $2a$, of a regular polygon of n sides, and let C be the centre of the inscribed circle. Draw CA, CB, and draw CD at right angles to AB. Then D is the middle point of AB.

By geometry, angle $ACD = \frac{1}{2} \angle ACB = \frac{1}{2} \cdot \frac{360°}{n} = \frac{180°}{n}$.

Also, by geometry,

$$\left[\text{angle } DAC = \tfrac{1}{2}\left(\frac{2n-4}{n}\right)90° = \left(\frac{n-2}{n}\right)90°. \right]$$

Hence, in the triangle ADC, the side AD and the angles are known; therefore CD, the radius of the circle inscribed in the polygon, can be found. On making similar constructions, the solution of the other problems referred to above will be apparent. The perpendicular from the centre of the circle to a side of the inscribed polygon is called the apothem of the polygon.

EXAMPLES.

1. The side of a regular heptagon is 14 ft.: find the radii of the inscribed and circumscribing circles; also, find the difference between the areas of the heptagon and the inscribed circle, and the difference between the area of the heptagon and the area of the circumscribing circle.

2. The side of a regular pentagon is 24 ft. Find quantities as in Ex. 1.

3. The side of a regular octagon is 24 ft. Find quantities as in Ex. 1.

4. The radius of a circle is 24 ft. Find the lengths of the sides and apothems of the inscribed regular triangle, quadrilateral, pentagon, hexagon, heptagon, and octagon. Compare the area of the circle and the areas of these regular polygons; also compare the perimeters of the polygons and the circumference of the circle.

5. For the same circle as in Ex. 4, find the lengths of the sides of the circumscribing regular figures named in Ex. 4. Compare their areas and perimeters with the area and circumference of the circle.

6. If a be the side of a regular polygon of n sides, show that R, the radius of the circumscribing circle, is equal to $\frac{1}{2}\, a \operatorname{cosec} \frac{180°}{n}$; and that r, the radius of the circle inscribed, is equal to $\frac{1}{2}\, a \cot \frac{180°}{n}$.

7. If r be the radius of a circle, show that the side of the regular inscribed polygon of n sides is $2r \sin \frac{180°}{n}$; and that the side of the regular circumscribing polygon is $2r \tan \frac{180°}{n}$.

8. If a be the side of a regular polygon of n sides, R the radius of the circumscribing circle, and r the radius of the circle inscribed, show that area of polygon $= \frac{1}{4} na^2 \cot \frac{180°}{n} = \frac{1}{2} nR^2 \sin \frac{360°}{n} = nr^2 \tan \frac{180°}{n}$.

34. Solution of oblique triangles. Since an oblique triangle can be divided into right-angled triangles by drawing a perpendicular from a vertex to the opposite side, it may be expected that knowledge concerning the solution of right-angled triangles will be of service in solving oblique triangles. This expectation will not be disappointed. An examination, which it is advisable for the student to make before proceeding farther, will show that all the sets of *data* from which a definite triangle can be drawn are those indicated in (1)–(4) below. *The ability to make the following geometrical constructions is presupposed:*

(1) To draw a triangle on being given two of its angles and a side opposite to one of them;

(2) To draw a triangle on being given two of its sides and an angle opposite to one of them;

(3) To draw a triangle on being given two of its sides and their included angle;

(4) To draw a triangle on being given its three sides.

In what follows, only the steps in the solutions will be indicated. The examples that are worked may be saved, so that the amount of labor required by the method of solution shown here can be compared with the amount required by another method which will be described later.

There are four cases in the solution of oblique triangles; these cases correspond to the four problems of construction stated above.

Case I. *Given two angles and a side opposite to one of them.*

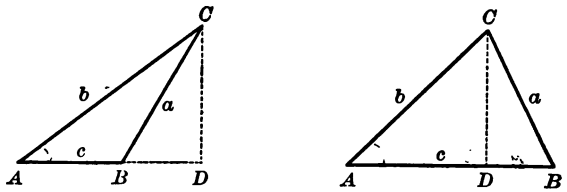

Fig. 30.

In ABC let A, B, a be known. Angle C and sides b, c are required. From C draw CD at right angles to AB or AB produced.

In triangle CBD, angle CBD and side CB are known. $\therefore$ BD and DC can be found.

Then, in triangle ACD, side DC and angle A are known. $\therefore$ AC and AD can be found.

Side $AB = AD - BD$ when B is obtuse, and $AB = AD + DB$ when B is acute.

$$\text{Angle } C = 180° - (A + B).$$

Another method of solution is given in Art. **55.**

EXAMPLES.

1. Ex. 1, Art. 55. **2.** Ex. 2, Art. 55.

3. Ex. 2, Art. 60. **4.** Other Exs. in Arts. 55, 60.

Case II. *Given two sides and an angle opposite to one of them.*

N.B. The first part of the text in Art. 56 should be read at this time.

Let (Fig. 30) AC, BC, angle A be known. [In a certain case, as shown in Art. 56, two triangles can be drawn which satisfy the given conditions.] From C draw CD at right angles to AB or AB produced.

In ACD, AC and A are known. $\therefore$ AD, DC, angle ACD, can be found.

Then, in BCD, BC and CD are known. $\therefore$ BD, angle DBC, can be found.

In one figure, $AB = AD - BD$, angle $ABC = 180° - CBD$.

In other figure, $AB = AD + DB$. In both figures,

$$\text{angle } ACB = 180° - (CAB + ABC).$$

Another method of solution is given in Art. 56.

EXAMPLES.

1. Ex. 1, Art. 56.
2. Ex. 2, Art. 56.
3. Ex. 1, Art. 60.
4. Other Exs. in Arts. 56, 60.

CASE III. *Given two sides and their included angle.*

In ABC let b, c, A be known. Side a, B, C are required. From C draw CD at right angles to AB or AB produced.

In ACD, AC and angle A are known. $\therefore$ CD and AD can be found.

Then, in triangle CDB, CD is now known, and $BD = AD - AB$ or $AB - AD$. $\therefore$ Angle CBD can be found. Angle $ABC = 180° - CBD$ in figure on the left. Angle $ACB = 180° - (A + B)$.

Another method of solution is given in Art. 57.

EXAMPLES.

1. Ex. 1, Art. 57.
2. Ex. 2, Art. 57.
3. Ex. 1, Art. 61.
4. Other Exs. in Arts. 57, 61.
5. Ex. in Art. 21.

CASE IV. *Given the three sides.*

In ABC let a, b, c be known. The angles A, B, C are required. From any vertex C draw CD at right angles to AB or AB produced.

$$CD^2 = b^2 - AD^2; \tag{1}$$

also,
$$CD^2 = a^2 - DB^2 = a^2 - (c - AD)^2.$$

$$\therefore b^2 - AD^2 = a^2 - (c - AD)^2.$$

$$\therefore AD = \frac{b^2 + c^2 - a^2}{2c}. \tag{2}$$

Also, $DB = AD - c$ (one figure), or $c - AD$ (other figure).

Hence, in ACD, AC, AD are known. $\therefore$ A can be found.

Also, in CDB, CB, DB are known. $\therefore$ B can be found.

$$C = 180° - (A + B).$$

Another method of solution is given in Art. 58.

EXAMPLES.

1. Ex. 1, Art. 58. **2.** Ex. 2, Art. 58.

3. Ex. 1, Art. 62. **4.** Other Exs. in Arts. 58, 62.

5. Solve some of the problems in Art. 63 by means of right-angled triangles.

34 *a*. The area of a triangle in terms of the sides. (See Fig. 30.) From (1), (2), Case IV., Art. 34,

$$CD^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2 = \frac{4c^2b^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{[2cb + b^2 + c^2 - a^2][2cb - (b^2 + c^2 - a^2)]}{4c^2}$$

$$= \frac{[(b + c)^2 - a^2][a^2 - (b - c)^2]}{4c^2}$$

$$= \frac{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{4c^2}.$$

Let $$a + b + c = 2s;$$

then $$2(s - a) = a + b + c - 2a = -a + b + c.$$

Similarly, $2(s - b) = a - b + c$; $2(s - c) = a + b - c$.

Then $$CD = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

$\therefore$ Area $ABC = \frac{1}{2}\, AB \cdot CD = \sqrt{s(s - a)(s - b)(s - c)}$.*

Ex. Find the areas of the triangles in Exs. Case IV., Art. 34. Check the results by finding the areas by the method of Art. 31.

---

* This is sometimes known as *Hero's Formula* for the area of a triangle. It was discovered by Hero (or Heron) of Alexandria, who lived about 125 B.C., and placed engineering and land surveying on a scientific basis.

34 b. Distance and dip of the visible horizon.

Let C be the centre of the earth, and let the radius be denoted by r.

Let P be a point above the earth's surface, and let its height PL be denoted by h.

Join P, C; draw PB from P to any point in the visible horizon; draw the horizontal line PH in the same plane with PC, PB. Then angle HPB is called the dip of the horizon. By geometry,

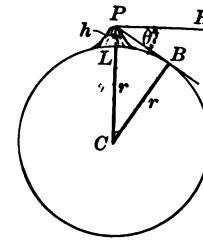

FIG. 30a.

$$\text{angle } PBC = 90°.$$

$$PB^2 = PC^2 - CB^2 = (r+h)^2 - r^2 = 2rh + h^2.$$

Since h^2 is very small compared with $2rh$,

$$PB = \sqrt{2rh} \text{ approximately.}$$

Take $r = 3960$ mi., and let h be measured in feet. Then $\frac{h}{5280}$ = height of P in miles.

$$\therefore PB = \sqrt{2 \times 3960 \times \frac{h}{5280}} \text{ mi.} = \sqrt{\tfrac{3}{2}h} \text{ mi.}$$

Hence, the distance of the horizon in miles is approximately equal to the square root of one and one-half times the height in feet.

EXAMPLES.

1. A man whose eye is 6 ft. from the ground is standing on the seashore. How far distant is his horizon?

$$\text{Distance} = \sqrt{\tfrac{3}{2} \times 6} \text{ mi.} = 3 \text{ mi.}$$

2. Find the greatest distance at which the lamp of a lighthouse can be seen, the light being 80 ft. above the sea level.

3. Find the height of the lamp of a lighthouse above the sea level when it begins to be seen at a distance of 12 mi.

4. From the top of a cliff, 40 ft. above the sea level, the top of a steamer's funnel which is known to be 30 ft. above the water is just visible. What is the distance of the steamer?

5. Find the distance and dip of the horizon at the top of a mountain 3000 ft. high?

6. Find the distance and dip of the horizon at the top of a mountain $2\frac{1}{2}$ mi. high.

34 c. Examples in the measurement of land. In order to find the area of a piece of ground, a surveyor measures distances and angles sufficient to provide *data* for the computation. An account

of his method of doing this, and of his arrangement of the *data* and the results in a simple, clear, and convenient form, belongs to special works on surveying. This article merely gives some examples which can be solved without any knowledge of professional details. The various rules for finding the area, of a triangle and a trapezoid, are supposed to be known. In solving these problems, the student should make the plotting or mapping an important feature of his work.

The Gunter's chain is generally used in measuring land. It is 4 rods or 66 feet in length, and is divided into 100 links.

An acre = 10 square chains=4 roods=160 square rods or poles. The points of the compass have been explained in Art. 30.

EXAMPLES.

1. A surveyor starting from a point A runs S. 70° E. 20 chains, thence N. 10° W. 20 chains, thence N. 70° W. 10 chains, thence S. 20° W. 17.32 chains to the place of beginning. What is the area of the field which he has gone around?

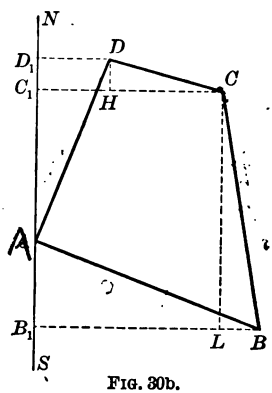

FIG. 30b.

Make a plot or map of the field, namely, $ABCD$. Here, AB represents 20 chains, and the bearing of B from A is S. 70° E. BC represents 20 chains, and the bearing C from B is N. 10° W., and so on. Through the most westerly point of the field draw a north-and-south line. This line is called *the meridian.* In the case of each line measured, find the distance that one end of the line is east or west from the other end. This easting or westing is called *the departure of the line.* Also find the distance that one end of the line is north or south of the other end. This northing or southing is called *the latitude of the line.* For example, in Fig. 30 *b*, the departures of AB, BC, CD, DA, are B_1B, BL, CH, DD_1, respectively; the latitudes of the boundary lines are AB_1, B_1C_1, C_1D_1, D_1A, respectively. It should be observed (Art. 36) that the algebraic sum of the departures of the boundary lines is zero, and so also is the algebraic sum of their latitudes. The following formulas are easily deduced:

Departure of a line = length of line × sine of the bearing;

Latitude of a line = length of line × cosine of the bearing.

By means of the departures, *the meridian distance of a point* (*i.e.* its distance from the north-and-south line) can be found. Thus the meridian

distance of C is C_1C, and $C_1C = D_1D + HC$. Hence in Fig. 30 b, AB_1, B_1B, B_1C_1, C_1C, C_1D_1, D_1D can be computed. Now

$$\text{area } ABCD = \text{trapezoid } D_1DCC_1 + \text{trapezoid } C_1CBB_1 - \text{triangle } ADD_1 - \text{triangle } ABB_1.$$

The areas in the second member can be computed; it will be found that area $ABCD = 26$ acres.

NOTE. Sometimes the bearing and length of one of the lines enclosing the area is also required. These can be computed by means of the latitudes and departures of the given lines. The formulation of a simple rule for doing this is left as an exercise to the student.

2. In Ex. 1, deduce the length and bearing of DA from the lengths and bearings of AB, BC, CD.

3. A surveyor starts from A and runs 4 chains S. 45° E. to B, thence 5 chains E. to C, thence 6 chains N. 40° E. to D. Find the distance and bearing of A from D; also, the area of the field $ABCD$. Verify the results by going around the field in the reverse direction, and calculating the length and bearing of BA from the lengths and directions of AD, DC, CB.

4. A surveyor starts from one corner of a pentagonal field, and runs N. 25° E. 433 ft., thence N. 76° 55′ E. 191 ft., thence S. 6° 41′ W. 539 ft., thence S. 25° W. 40 ft., thence N. 65° W. 320 ft. Find the area of the field. Deduce the length and direction of one of the sides from the lengths and directions of the other four.

5. From a station within a hexagonal field the distances of each of its corners were measured, and also their bearings; required its plan and area, the distances in chains and the bearings of the corners being as follows: 7.08 N.E., 9.57 N. ½ E., 7.83 N.W. by W., 8.25 S.W. by S., 4.06 S.S.E. 7° E., 5.89 E. by S. 3½° E.

35. Summary. Chapter II. was concerned with defining and investigating certain ratios inseparably connected with (acute) angles, and attention was directed to the tables of these ratios and their logarithms. In Chap. III. it was shown how these definitions and tables can be used in finding parts of a right-angled triangle when certain parts are known. In Chap. IV. the knowledge gained in Chap. III. was employed in the solution of some of the many problems in which right-angled triangles appear. In Art. 34 it has been seen that this knowledge can serve for the solution of oblique triangles. It follows, then, that it can serve for the solution of problems in which oblique triangles appear, and, accordingly, for the solution of *all problems*

involving the measurement of straight lines only. Consequently, the student is now able, without any additional knowledge of trigonometry, to solve the numerical problems in Chaps. VII., VIII. It is thus apparent that even a slight acquaintance with the ratios defined in Chap. II. has greatly increased the learner's ability to solve useful practical problems.

Oblique triangles can sometimes be solved in a more elegant manner than that pointed out in Art. 34. In order to show this, further consideration of angles and the trigonometric ratios is necessary. Consequently, in Chap. V. some important additions are made to the idea of a straight line and the idea of an angle; the *trigonometric ratios are defined in a more general way*, namely, for all angles, instead of for acute angles only, and *the principal relations of these ratios are deduced.* Chapter VI. treats of *the ratios of two angles in combination.* While it is necessary to consider these matters before proceeding to the solution of oblique triangles given in Chap. VII., it should be said that the knowledge that will be gained in Chaps. V., VI., VII., is necessary and important for other purposes besides the solution of triangles. In fact, the latter is one of the least important of the results obtained in these chapters.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chap. IV. will be found at page* 184.

CHAPTER V.

TRIGONOMETRIC RATIOS OF ANGLES IN GENERAL.

36. Directed lines. Let MN be a line unlimited in length in the directions of both M and N. Suppose that a point starts at P and moves along this line for some given distance. In order to mark where the point stops, it is necessary to know, not only this distance, but also the *direction* in which the point has moved from P. This direction may be indicated in various ways; by saying, for instance, that the point moves *toward the right* from P, or *toward the left* from P; that the point moves *toward* N, or *toward* M; that the point moves *in the direction of* N, or *in the direction of* M; and so on. Mathematicians, engineers, and others

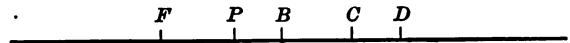

FIG. 31.

have agreed to use a particular method (and this practically comes to the adoption of a *particular rule*) for indicating the two opposite directions in which a point can move along a line, or in which distances along a line can be measured. This *convention*, or rule which has been adopted for the sake of convenience, is as follows:

Distances measured along a line, or along parallel lines, in one direction shall be called ***positive*** *distances, and shall be denoted by the sign* $+$*; distances measured in the opposite direction shall be called* ***negative*** *distances, and shall be denoted by the sign* $-$.

The convenience of this custom, fashion, or rule, will become apparent in the examples that follow.* In Fig. 31 let distances

* Advances in mathematics have often depended upon the introduction of a good custom which has at last been universally adopted and made a rule. Thus, for example, the custom of using exponents to show the power to which a quantity is raised, which was first introduced in the first half of the sixteenth century, and made gradual progress until its final establishment in the latter half of the seventeenth century, has been of great service in aiding the advances of algebra.

measured in the direction of N be taken *positively;* then distances measured in the direction of M will be taken *negatively.* On directed lines the direction in which a line is measured, or in which a point moves on a line, is indicated by *the order of the letters* naming the line. Thus, for example, if a point moves from B to C, the distance passed over is read BC. In this reading, the starting point is indicated by the first letter B, and the stopping point, by the last letter C. After the same fashion, CB means the distance from C to B. If, for instance, there are 3 units of length between B and C, then $BC = +3$, $CB = -3$.

EXAMPLES.

1. Suppose a point (Fig. 31) moves from P to B, thence to C, thence to D, thence to F. Let the number of units of length between P and B, B and C, C and D, F and D, be 2, 3, 2, 10, respectively. The point starts at P and stops at F; hence the distance from the starting point to the stopping point is PF. In this case the point's trip from P to F is made in several steps as indicated above. That is, on properly indicating the lines passed over,

$$\begin{aligned} PF &= PB + BC + CD + DF \\ &= 2 + 3 + 2 - 10 \quad [\because FD = +10, \text{ then } DF = -10.] \\ &= -3. \end{aligned}$$

This shows that the final position of the moving point is three units to the left of P. This example also shows one great *convenience of the rule of signs in measurement*, namely, that by attending to this rule and to the proper naming of the lines passed over by a moving point, *one immediately obtains the result of the successive movements.*

NOTE. In the following examples, in lines that lie east and west, let measurements toward the east be taken positively; in lines that lie north and south, let measurements toward the north be taken positively.

2. A man travelling on an east and west line goes east 20 mi., then east 16 mi., then west 18 mi., then east 30 mi. What is his final distance from the starting point? [Draw a figure, and indicate the successive trips by letters.]

3. A man travelling on an east and west line goes west 20 mi., then east 10 mi., then east 25 mi., then east 30 mi., then west 45 mi. Do as in Ex. 2.

4. A man travelling on a north and south line goes north 100 mi., then south 60 mi., then south 110 mi., then north 200 mi., then north 15 mi., then south 247 mi. Do as in Ex. 2.

37. Trigonometric definition of an angle. Angles unlimited in magnitude. Positive and negative angles. In books on plane geometry a plane angle is defined in various ways, namely, as the inclination of two lines to one another, which meet together, but are not in the same direction; or, as the figure formed by two straight lines drawn from the same point; or, as the amount of divergence of two lines which meet in a point, or would meet if produced; or, as the opening between two straight lines which meet; or, as the difference in direction of two lines which meet; and so on. In these definitions an angle is always regarded as less than two right angles. A definition according to which angles are less restricted, is adopted in trigonometry.

Trigonometric definition of an angle. The angle between two lines which intersect is the **amount of turning** which a line revolving about their point of intersection makes, when it begins its revolution at the position of one of the two lines and stops in the position of the other line. Thus, for example, the angle between OX and OQ is the amount of turning which is made by a line OP revolving about O when OP starts revolving from the position QX and stops its revolution at the position OQ. The line OX at which the revolution begins, is called the **initial line**; the line OQ at which the revolution ends, is called the **terminal line**; when the turning line OP has reached the terminal position OQ, OP is said to have *described the angle* XOQ.

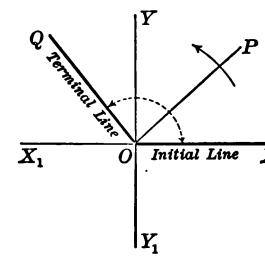

FIG. 32.

Let YOY_1 be at right angles to X_1OX. When OP has revolved until it lies in the position OY, it has described a right angle, or 90°; when it has revolved until it lies in the position OX_1, it has described two right angles, or 180° (this is usually termed "a straight angle" or "a flat angle"); when OP keeps on turning until it is in the position OY_1, it has described three right angles, or 270°; when OP has again reached the position OX, that is, when it has made one complete revolution, it has described four right angles, or 360°.

Angles unlimited in magnitude. Now OP may start revolving from OX, make one complete revolution, continue to revolve, and *then* cease revolving when it has again reached the position OQ. This is indicated in Fig. 33. Or, OP may make two complete revolutions before it comes to rest in the position OQ; or, it may make three revolutions, or four, or as many as one please, before ceasing its revolution at the position OQ. An angle of 360° is described each time that OP makes a complete revolution, and OP can make as many revolutions as one please. According to the trigonometric definition of an angle, therefore, **angles are unlimited in magnitude.**

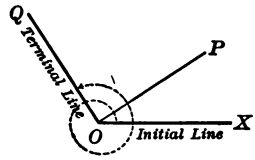

FIG. 33.

Moreover, when this definition of an angle is adopted, **the same figure can represent an infinite number of different angles.** Any two of these angles differ from each other by a whole number of complete revolutions. For instance, Fig 34 may represent 60°, 360°+60° or 420°, $2 \cdot 360° + 60°$ or 780°, $3 \cdot 360° + 60°$ or 1140°, $\cdots$, $n \cdot 360° + 60°$, in which n denotes any whole number. *Any two of these angles differ by a multiple of* 360°. Angles which have the same initial and terminal lines may be called **coterminal angles.**

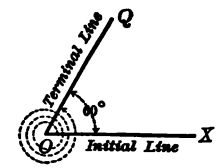

FIG. 34.

Positive and negative angles. The revolving line OP (Fig. 32) may revolve about O in the same direction as that in which the hands of a watch revolve, or it may revolve in the opposite direction. The following *convention* (see Art. 36) has been adopted for the sake of distinguishing these two opposite directions:

When the turning line revolves in a **counter-clockwise direction,** *the angles described are said to be* **positive,** *and are given the* ***plus*** *sign; when the turning line revolves in a* **clockwise direction,** *the angles described are said to be* **negative,** *and are given the* ***minus*** *sign.*

Thus, for example, Fig. 34 represents the angles $+60°$, $-300°$; further, this figure represents the angles $60° \pm n \cdot 360°$, in which n denotes any whole number. The angle $-300°$ is included in these angles, for, on putting -1 for n, there is obtained $60°-360°$, *i.e.* $-300°$. (Negative angles are also unlimited in magnitude.)

As in the case of lines, *the sign of an angle can be denoted by the order of the letters used in naming the angle.* Thus XOQ denotes the angle formed by revolving OX toward OQ, and QOX denotes the angle formed by revolving OQ toward OX. Accordingly, $QOX = -XOQ$.

Quadrants. In Fig. 32, XOY, YOX_1, X_1OY_1, Y_1OX, are called *the first, second, third, and fourth quadrants,* respectively. When the turning line ceases its revolution at some position between OX and OY, the angle described is said to be *an angle in the first quadrant;* when the final position of the turning line is between OY and OX_1, the angle described is said to be *in the second quadrant;* and so on for the third and fourth quadrants.

For example, the angles 30°, − 345°, 395°, 725° are all in the first quadrant; the angles − 60°, 340°, 710° are all in the fourth quadrant; the angle − 225° is in the second quadrant, and the angle 225° is in the third quadrant.

NOTE. While all acute angles are in the first quadrant, not all angles which are in the first quadrant are acute.

EXAMPLES.

NOTE. When it is necessary, the number of revolutions and their direction may be indicated on the figure in the manner shown in Fig. 34.

Lay off the following angles with the protractor: In the case of each angle name the least positive angle that has the same terminal line. Name the quadrants in which the angles are situated. In the case of each angle name the four smallest positive angles that have the same terminal line.

1. 137°, 785°, 321°, 930°, 840°, 1060°, 1720°, 543°, 3657°.

2. − 240°, − 337°, − 967°, − 830°, − 750°, − 1050°, − 7283°.

3. − 47° + 230° + 37°, 420° − 470° + 210° − 150°, 230° − 47° + 37°, 230° + 37° − 47°.

38. Supplement and complement of an angle. *The supplement of an angle* is that angle which must be *added to it* in order to make two right angles, or 180°; *the complement of an angle* is that angle which must be *added to it* in order to make one right angle, or 90°. Thus, if A be *any* angle, then

$$\text{supplement of angle } A = 180° - A,$$

$$\text{complement of angle } A = 90° - A.$$

EXAMPLES.

1. What are the complements and supplements of 40°, 227°, − 40°?

complement of $40° = 90° - 40° = 50°$;
supplement of $40° = 180° - 40° = 140°$.

complement of $227° = 90° - 227° = -137°$;
supplement of $227° = 180° - 227° = -47°$.

complement of $-40° = 90° - (-40°) = 130°$;
supplement of $-40° = 180° - (-40°) = 220°$.

2. By means of a figure verify the results obtained in Ex. 1.

3. What are the complements of − 230°, 150°, − 40°, 340°, 75°, 83°, 12°, − 295°, − 324°, 200°, 240°, − 110°, − 167°?

4. What are the supplements of the angles in Ex. 3?

5. Verify the results in (3), (4), by drawing figures.

39. The convention of signs on a plane. Articles 36, 37 contain statements of the conventions adopted regarding *the algebraic signs to be given to distances* measured on parallel straight lines, *and to angles* described by the revolution of a turning line. A figure, such as Figs. 32, 35, will be frequently used in the articles that follow. In this figure, OX is the initial line, the turning line revolves about O, and YOY_1 is at right angles to X_1OX. *The following convention has been adopted regarding the lines which will be used:*

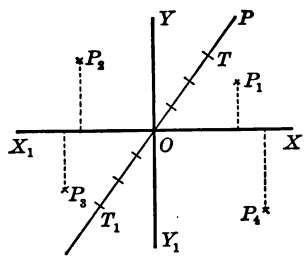

FIG. 35.

Horizontal lines measured in the direction of X are taken **positively**;
Horizontal lines measured in the direction of X_1 are taken **negatively**;
Vertical lines measured upward are taken positively;
Vertical lines measured downward are taken negatively.

The distance of points, such as P_1, P_2, P_3, P_4, from X_1X, is always measured *from* X_1X *toward* the points.

Any turning line (or oblique line) as OP is measured positively

from O toward the *end* of the turning line which lies in the direction of X from O when the turning line coincides with the initial line. Thus a distance $+3$ on OP will terminate at T, distant 3 units from O, and a distance -3 on OP will terminate at T_1, distant 3 units from O, but in the direction opposite to the former. This is sometimes briefly expressed in the words: *the turning line carries its positive direction with it in its revolution.*

40. General definition of the trigonometric ratios. The remarks in this article apply to each of the four figures below. In each figure, O is the point about which the angle is described, OX is the initial line, and OP is the terminal line. The first figure represents any angle in the first quadrant; the second figure represents any angle in the second quadrant; the third figure, any angle in the third quadrant; and the fourth figure, any angle in the fourth quadrant. In each figure the angle will be called A.

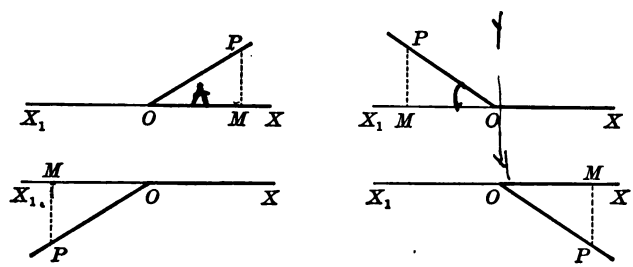

FIG. 36.

Let P be any point in OP, the terminal line of any angle A. From P draw PM at right angles to the initial line OX, or to the initial line produced in the negative direction. In each figure, **OM** is the distance measured along X_1OX *from* the point O *to* the foot of the perpendicular MP, and **MP** is the distance *from* X_1OX *to* the point P. Following are the definitions of the trigonometric ratios; these definitions apply to the angles represented in Fig. 36, and, accordingly, to all angles whatsoever. [*Particular attention should be paid to the order of the letters used in naming the lines, for this order indicates the direction in which the line is measured.* See Art. 36.]

The ratio $\frac{MP}{OP}$ is called the *sine* of the angle A.

The ratio $\frac{OM}{OP}$ is called the *cosine* of the angle A.

The ratio $\frac{MP}{OM}$ is called the *tangent* of the angle A.

The ratio $\frac{OM}{MP}$ is called the *cotangent* of the angle A.

The ratio $\frac{OP}{OM}$ is called the *secant* of the angle A.

The ratio $\frac{OP}{MP}$ is called the *cosecant* of the angle A.

These definitions may be briefly stated:

$$\left.\begin{array}{lll} \sin A = \frac{MP}{OP}. & \tan A = \frac{MP}{OM}. & \sec A = \frac{OP}{OM}. \\ \cos A = \frac{OM}{OP}. & \cot A = \frac{OM}{MP}. & \operatorname{cosec} A = \frac{OP}{MP}. \end{array}\right\} \quad (1)$$

Inspection will show that the definitions of the trigonometric ratios for acute angles given in Art. 12, are in accordance with these general definitions.

N.B. The *projection* definitions of the trigonometric ratios are given in Note B, Appendix.

41. The algebraic signs of the trigonometric ratios for angles in the different quadrants. Figures 36 show that if the angle A is in the first, second, third, fourth quadrants, then the algebraic sign of MP is $+$, $+$, $-$, $-$, respectively, and the algebraic sign of OM is $+$, $-$, $-$, $+$, respectively. As stated in Art. 39, OP is always taken positively. Hence, on paying regard to the algebraic signs of OM, MP, OP, in the several quadrants, it will be seen that the ratios of the angles in these quadrants are positive or negative, as indicated in the following table:

QUADRANT.	I.	II.	III.	IV.
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−
Cotangent	+	−	+	−
Secant	+	−	−	+
Cosecant	+	+	−	−

EXAMPLES.

The student is advised to preserve his work on these examples. If he regards his results attentively, he will probably discover some useful facts, and *be able to deduce some useful theorems*, concerning angles in general. Any preceding results, such as those in Art. 15, may be used as an aid in solving these exercises.

1. Find the ratios of 945°.

$$945° = 2 \times 360° + 225°.$$

$\therefore$ OP, the terminal line of angle 945°, has the position shown in Fig. 37. For this position of the terminal line, OM and MP are both negative. As shown in Art. 15, the lines OM, MP, OP, in this figure are respectively proportional to 1, 1, $\sqrt{2}$. These are indicated on the figure *with their proper algebraic signs*. It is immediately apparent that

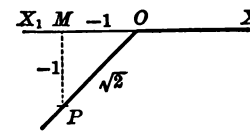

FIG. 37.

$$\sin 945° = -\frac{1}{\sqrt{2}}, \quad \cos 945° = -\frac{1}{\sqrt{2}}, \quad \tan 945° = +1, \quad \cot 945° = +1,$$

$$\sec 945° = -\sqrt{2}, \quad \operatorname{cosec} 945° = -\sqrt{2}.$$

2. Construct, and find the ratios of, 420°, 780°, 1140°.

3. Construct, and find the ratios of, 120°, 480°, 240°, 600°, − 60°, 300°, 660°, − 720°.

4. Construct, and find the ratios of, 150°, 410°, 210°, −150°, 330°, −390°.

5. Construct, and find the ratios of, 45°, 765°, 135°, − 225°, 225°, 585°, − 405°, 1035°.

6. Construct, and find the ratios of, − 754°, 487°, − 245°.

7. Compare the ratios of 90° − 30°, 90° − 60°, 90° − 45°, 90° − 135°, 90° − 240°, 90° − 300°, with the ratios of 30°, 60°, 45°, 135°, 240°, 300°, respectively.

8. Compare the ratios of $90° + 30°$, $90° + 60°$, $90° + 45°$, $90° + 135°$, $90° + 240°$, $90° + 300°$, with the ratios of $30°$, $60°$, $45°$, $135°$, $240°$, $300°$, respectively.

9. Compare the ratios of $180° - 30°$, $180° - 60°$, $180° - 45°$, $180° - 135°$, $180° - 240°$, $180° - 300°$, with the ratios of $30°$, $60°$, $45°$, $135°$, $240°$, $300°$, respectively. So, also, the ratios of $-30°$, $-60°$, $-45°$, etc.

10. Are any general relations *indicated* by the results of Exs. 7, 8, 9? If so, state these relations. Try to prove them.

42. To represent the angles geometrically when the ratios are given. In constructing the angles in this article it is necessary to bear in mind that, according to the definitions given in Art. 40:

When MP is *positive*, it can be drawn in the *first and second quadrants;*

When MP is *negative*, it can be drawn in the *third and fourth quadrants;*

When OM is *positive*, it is to be drawn in the direction OX;

When OM is *negative*, it is to be drawn in the direction OX_1;

and OP is to be taken positively.

EXAMPLES.

1. Represent by a figure the angles which have sines equal to $\frac{3}{4}$. Calculate their other ratios. Let A denote an angle whose sine is $\frac{3}{4}$; *i.e.* let $\sin A = \frac{3}{4}$. But $\sin A = \frac{MP}{OP}$ (Art. 40). Hence, $MP : OP = 3 : 4$; and if $OP = 4$, then $MP = 3$. Now, MP can be drawn positively in both the first and second quadrants. Hence the problem amounts to finding a point in the first quadrant and a point in the second quadrant, each at a distance 4 from O and a distance 3 from X_1OX. The result is indicated in Fig. 38. The student can make the construction for himself. The angles having sines equal to $\frac{3}{4}$, accordingly, include *all the angles* which have OP for a terminal line, *and all the angles* which have OP_1 for a terminal line. By Art. 37 each of these two sets of angles consists of an infinite number of angles, any two of which differ from one another by a whole number, positive or negative, of complete revolutions. A general algebraic expression which includes all these angles is deduced in Art. 85.

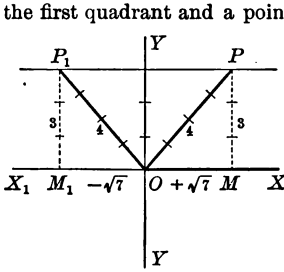

Fig. 38.

Figure 38 shows that for angles having OP for a terminal line, cosine is $\frac{\sqrt{7}}{4}$, tangent is $\frac{3}{\sqrt{7}}$, etc.; and that for angles having OP_1 for a terminal line, cosine is $\frac{-\sqrt{7}}{4}$, tangent is $-\frac{3}{\sqrt{7}}$, etc. Since the given sine is positive, it is apparent that the angles required, must be in the first and second quadrants. (See Art. 41.)

2. Represent by a figure all the angles which have tangents equal to $-\frac{3}{4}$. Let A denote an angle whose tangent is $-\frac{3}{4}$; *i.e.* let $\tan A = -\frac{3}{4}$. [This may be written, $\frac{+3}{-4}$ or $\frac{-3}{+4}$.] But $\tan A = \frac{MP}{OM}$ (Art. 40). Hence, if $MP = 3$, then $OM = -4$, and if $MP = -3$, then $OM = 4$. When MP is positive and OM is negative, OP can lie only in the second quadrant. When MP is negative and OM is positive, OP can lie only in the fourth quadrant. Figure 39 represents the angles. The student can make the construction for himself. Thus, the angles whose tangents are equal to $-\frac{3}{4}$, consist of the set of angles, infinite in number, which have OP for a terminal line, and the set of angles, infinite in number, which have OP_1 for a terminal line.

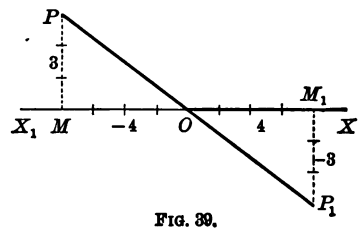

FIG. 39.

3. Calculate the other ratios of the angles in Ex. 2.

4. Represent geometrically all the angles whose cosine is $\frac{4}{5}$. Calculate their other ratios.

5. So, also, when the cosine is $-\frac{4}{5}$.

6. So, also, when the tangent is $\frac{4}{3}$. [Note. $\frac{4}{3} = \frac{+4}{+3} = \frac{-4}{-3}$.]

7. So, also, when the sine is $-\frac{4}{5}$.

8. So, also, when the secant is $\frac{5}{3}$.

9. So, also, when the secant is $-\frac{4}{3}$.

10. So, also, when the cosecant is -2; $\frac{7}{4}$.

N.B. The student is now strongly recommended to delay the reading of the next article until after he has reviewed the properties stated in Art. 13, and, if possible, determined what are *the correct corresponding statements for angles in general.*

43. Connection between angles and the trigonometric ratios. For the same terminal position of the revolving line OP each of the ratios, $\frac{MP}{OP}$, etc., in (1) Art. 40, is always the same, no matter

where the point P is taken on the revolving line. Thus, for example, let any other point P_1 (Fig. 40 a) be taken on the ter-

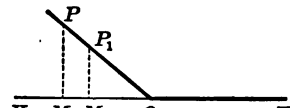

FIG. 40a.

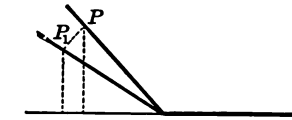

FIG. 40b.

minal line, and let P_1M_1 be drawn perpendicular to X_1OX. Then $\frac{MP}{OP} = \frac{M_1P_1}{OP_1}$. That is, the sine of any angle whose terminal line is OP, has a fixed definite value. The same can be shown for the other ratios. Hence, as already shown in Art. 13 for angles between 0° and 90°,

(1) *To each angle there corresponds but one value of each trigonometric ratio.*

Now let OP revolve a little from OP into the position OP_1 (Fig. 40 b). For convenience keep OP_1 equal to OP. Draw P_1M_1 at right angles to X_1OX. Then,

$$\text{the sine of the angle whose terminal line is } OP = \frac{MP}{OP},$$

$$\text{and the sine of the angle whose terminal line is } OP_1 = \frac{M_1P_1}{OP_1}.$$

Since M_1P_1 is not equal to MP, it follows that these two sines are unequal. Hence, the sine of an angle changes when the angle changes. The same can be shown for the other ratios. Hence,

(2) *The ratios of an angle change when the angle changes.*

The variation in the ratios as the angle increases, is discussed in Art. 77.

Ex. 1. In the above, OP_1 is taken equal to OP. Why does this not affect the generality of the deduction?

Ex. 2. Trace the changes in the trigonometric ratios as the turning line revolves from 0° to 360°. Compare your results with those of Art. 15 C, and those given in the table at the end of Art. 77.

It has been shown in Art. 37 that an infinite number of angles have the same terminal line. It follows that in each of the

figures in Art. 40, $\frac{MP}{OP}$ is the sine of an infinite number of angles. The same is true in the case of the other ratios. Moreover, the geometrical solutions in Art. 42 show that there are *two sets of angles corresponding to each given ratio,* and that *each set is infinite in number,* and has a particular terminal line. Hence,

(3) *To each value of a trigonometric ratio there corresponds an infinite number of angles.*

NOTE. The student will see, by turning to Arts. 84–87, that all angles which have the same sine, can be given in a simple formula, and that the same fact is true in the case of each of the other ratios. The deduction of these formulas, while easily possible at this place, is postponed in order to permit the early completion of the solution of triangles.

44. Relations between the trigonometric ratios of an angle. The relations between the trigonometric ratios of *an acute angle* were set forth in Art. 18. It will now be shown that these relations also hold for the ratios *of any angle.*

A. Inspection of the definitions (1), Art. 40, shows *the reciprocal relations,* namely:

$$\sin A \operatorname{cosec} A = 1; \quad \cos A \sec A = 1; \quad \tan A \cot A = 1. \qquad (1)$$

B. In each of the figures in Art. 40,

$$\tan A = \frac{MP}{OM} = \frac{\frac{MP}{OP}}{\frac{OM}{OP}} = \frac{\sin A}{\cos A}; \quad \cot A = \frac{OM}{MP} = \frac{\frac{OM}{OP}}{\frac{MP}{OP}} = \frac{\cos A}{\sin A}. \qquad (2)$$

C. In each of the figures in Art. 40,

$$\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2.$$

On dividing both members of this equation by $\overline{OP}^2$, $\overline{OM}^2$, $\overline{MP}^2$, in turn, and following the same process as that adopted in Art. 18, it results that

$$\sin^2 A + \cos^2 A = 1; \quad \sec^2 A = 1 + \tan^2 A; \quad \operatorname{cosec}^2 A = 1 + \cot^2 A. \qquad (3)$$

From the first of relations (3) it follows that

$$\cos A = \pm \sqrt{1 - \sin^2 A}.$$

This shows that, corresponding to *a given sine*, there are *two cosines* which are numerically equal, and opposite in algebraic sign. Ex. 1, Art. 42, illustrates this. This is also manifest in the table of signs in Art. 41. As indicated in this table, the sine is positive in the first and second quadrants, and then the cosine is positive and negative, respectively; the sine is negative in the third and fourth quadrants, and then the cosine is negative and positive, respectively. The signs of the remaining ratios corresponding to a given sine will be apparent on a short geometrical inspection, or by a glance at this table of signs. *When any single ratio is given, there is an ambiguity as to the signs of some of the other ratios.* Thus, to take another instance, it follows from the second of (3) that

$$\tan A = \pm\sqrt{\sec^2 A - 1}.$$

The secant of A is positive in the first and fourth quadrants, and then the tangent is positive and negative respectively; the secant of A is negative in the second and third quadrants, and then the tangent is negative and positive respectively. The double sign which appears in these relations was referred to in the examples, Art. 18. The student is advised to review and work the examples, for angles in general, in Art. 18.

EXAMPLES.

1. Given that $\sin A = \frac{3}{4}$; find the other ratios of A by means of the relations shown in this article.

[In Ex. 1, Art. 42, this problem is solved *geometrically;* here it will be solved *algebraically.*]

$$\cos A = \pm\sqrt{1 - \sin^2 A} = \frac{\pm\sqrt{7}}{4}; \quad \sec A = \frac{1}{\cos A} = \frac{4}{\pm\sqrt{7}};$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{4}{3}; \quad \tan A = \frac{\sin A}{\cos A} = \frac{3}{\pm\sqrt{7}}; \quad \cot A = \frac{1}{\tan A} = \frac{\pm\sqrt{7}}{3}.$$

Since the given sine is positive, the corresponding angles are in the first and second quadrants. Hence the double values of the calculated ratios are paired as follows:

$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\operatorname{cosec} A$
$\frac{3}{4}$	$\frac{+\sqrt{7}}{4}$	$\frac{3}{\sqrt{7}}$	$\frac{\sqrt{7}}{3}$	$\frac{4}{\sqrt{7}}$	$\frac{4}{3}$
$\frac{3}{4}$	$\frac{-\sqrt{7}}{4}$	$-\frac{3}{\sqrt{7}}$	$\frac{-\sqrt{7}}{3}$	$\frac{4}{\sqrt{7}}$	$\frac{4}{3}$

Find the other ratios *algebraically*, and verify the results *geometrically*, when:

2. $\cos A = -\frac{2}{3}$. 3. $\tan A = \frac{5}{7}$. 4. $\sec A = 4$. 5. $\operatorname{cosec} A = -5$.
6. $\sin A = -\frac{1}{7}$. 7. $\cos A = \frac{1}{4}$. 8. $\tan A = -3$. 9. $\cot A = \frac{2}{5}$.

Find the other ratios algebraically, and verify the results geometrically, when angle A satisfies the following *pairs of conditions:*

10. $\sin A = \frac{1}{2}$ and $\tan A = -\frac{1}{\sqrt{3}}$. 11. $\tan A = \sqrt{3}$ and $\sec A = -2$.
12. $\cos A = -\frac{2}{3}$ and $\sin A = +\frac{\sqrt{5}}{3}$. 13. $\sin A = -\frac{3}{5}$ and $\tan A = \frac{3}{4}$.

Give a geometrical solution of the following trigonometric equations:

14. $\sin A = \frac{2}{3}$. 15. $\cos A = -\frac{3}{7}$. 16. $\tan A = 4$. 17. $\sec A = 5$.

Name the four least angles, and also the four least positive angles, that satisfy the equations:

18. $\sin\theta = \frac{1}{\sqrt{2}}$. 19. $\tan\theta = \sqrt{3}$. 20. $\cos\theta = -\frac{1}{\sqrt{2}}$. 21. $\cot\theta = -\sqrt{3}$.

45. Ratios of $90° - A$, $180° - A$, $90° + A$, $-A$, compared with the ratios of A, A being any angle. The student may have suspected, from Exs. 7–10, Art. 41, that there is a close connection between the ratios of an angle A on the one hand, and the ratios of the angle $-A$ and of angles differing from A and $-A$ by multiples of 90°, on the other. He may have discovered already what the connection is. This connection, which is set forth in this article, is interesting in the study of angles, and has an important bearing on the construction of trigonometric tables, and on the solution of triangles.

In each figure in this article OP is the terminal line of the angle A, and OP_1 is the terminal line of the related angle which is under consideration; for the purpose of easy comparison, OP_1 is always taken equal to OP; MP, M_1P_1, are the perpendiculars drawn from the initial line to P, P_1, respectively. The deductions

made in the simplest case, namely, when A is an angle in the first quadrant, are true for all angles. The student is advised to consider only the simplest case, when first he considers the subject of this article, and then to draw the figures and make the deductions for himself, in the cases in which angle A is in the second, third, and fourth quadrants, respectively.

Note. A compound angle, $90° - A$, for instance, can be described by revolving the turning line *forward* through 90°, and then *backward* through an angle equal to A; or, these steps may be taken in a reverse order, namely, by revolving the turning line *backward* through an angle equal to A, and then *forward* through 90°. Similarly, for the compound angles $90° + A$, $180° \pm A$, etc.

For the sake of clearness of construction, it is better not to take the terminal line of A nearly midway between X_1OX and Y_1OY.

A. **Ratios of $90° - A$.** Describe the angles A, $90° - A$. Let OP, OP_1, be the terminal lines of A, $90° - A$, respectively. In Figs.

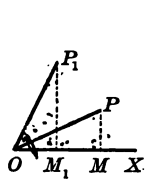

Fig. 41a.

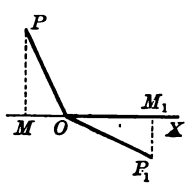

Fig. 41b.

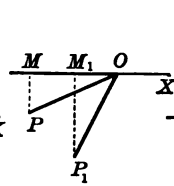

Fig. 41c.

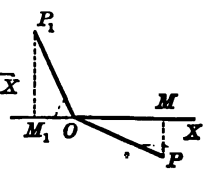

Fig. 41d.

41 *a*, 41 *b*, 41 *c*, 41 *d*, A is an angle in the first, second, third, and fourth quadrants, respectively.

Take OP_1 equal to OP, and draw MP, M_1P_1, at right angles to the initial line. In the triangles M_1OP_1, MOP, (in each figure) the angles at M_1, M, are right angles, angle M_1OP_1 = angle OPM, and $OP_1 = OP$. Hence these two triangles are equal, and

$$OM_1 = MP,\ M_1P_1 = OM.$$

The figures also show that, for A in each quadrant, OM_1, MP have the same algebraic sign, and M_1P_1, OM have the same sign. Hence, for all angles A,

$$\sin (90° - A) = \frac{M_1P_1}{OP_1} = \frac{OM}{OP} = \cos A;$$

$$\cos (90° - A) = \frac{OM_1}{OP_1} = \frac{MP}{OP} = \sin A;$$

$$\tan (90° - A) = \frac{M_1P_1}{OM_1} = \frac{OM}{MP} = \cot A;$$

$$\cot (90° - A) = \frac{OM_1}{M_1P_1} = \frac{MP}{OM} = \tan A;$$

$$\sec (90° - A) = \frac{OP_1}{OM_1} = \frac{OP}{MP} = \operatorname{cosec} A;$$

$$\operatorname{cosec} (90° - A) = \frac{O_1P_1}{M_1P_1} = \frac{OP}{OM} = \sec A.$$

Hence, the ratio of any angle is the same as the co-ratio of its complement. Compare with Art. 16. The relations for tangent, secant, cotangent, cosecant, can also be deduced from those of sine and cosine by means of Art. 44 (1), (2). Thus, for example,

$$\tan (90° - A) = \frac{\sin (90° - A)}{\cos (90° - A)} = \frac{\cos A}{\sin A} = \cot A.$$

B. **Ratios of 180° − *A*.** Describe the angles A, $180° - A$. Let OP, OP_1 be the terminal lines of A and $180° - A$ respectively. In Figs. 42 *a*, 42 *b*, 42 *c*, 42 *d*, A is an angle in the first, second, third, fourth quadrants, respectively. Take OP_1 equal to OP, and draw MP, M_1P_1, at right angles to the initial line. In the two triangles OM_1P_1, OMP, in each figure, the angles at M_1, M, are

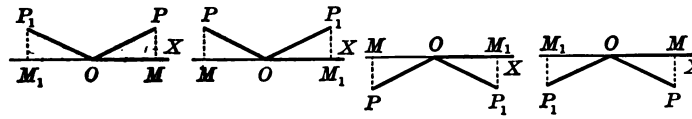

FIG. 42a. FIG. 42b. FIG. 42c. FIG. 42d.

right angles, angle $M_1OP_1 =$ angle MOP, $OP_1 = OP$. Hence, $OM_1 = OM$, and $M_1P_1 = MP$. In each figure, OM_1, OM have opposite algebraic signs, and M_1P_1, MP, have the same sign. Hence, **for all angles** A,

$$\sin (180° - A) = \frac{M_1P_1}{OP_1} = \frac{MP}{OP} = \sin A;$$

$$\cos (180° - A) = \frac{OM_1}{OP_1} = \frac{-OM}{OP} = -\cos A.$$

So also, $\tan(180° - A) = -\tan A$; $\cot(180° - A) = -\cot A$; $\sec(180° - A) = -\sec A$; $\operatorname{cosec}(180° - A) = \operatorname{cosec} A$.

The last four relations can be deduced by means of the figures, or by means of relations (1), (2), Art. 44. Hence, *any ratio of an angle is equal in magnitude to the same ratio of its supplement; the sines of supplementary angles have the same algebraic sign, and so have the cosecants; the other ratios of supplementary angles have opposite signs.*

C. **Ratios of $90° + A$.** Describe the angles A, $90° + A$. Let OP, OP_1, be the terminal lines of A, $90° + A$, respectively. In Figs. 43 *a*, *b*, *c*, *d*, A is an angle in the first, second, third, fourth quadrants, respectively. Take $OP_1 = OP$, and draw MP, M_1P_1, at right angles to the initial line. In the two triangles, OM_1P_1, OMP, (in each figure) the angles at M_1, M, are right angles,

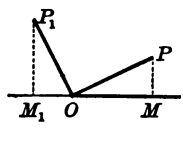

FIG. 43a.

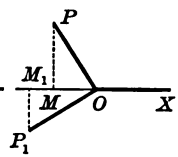

FIG. 43b.

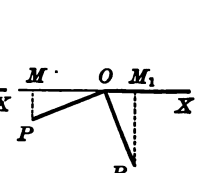

FIG. 43c.

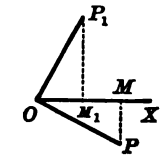

FIG. 43d.

angle M_1OP_1 = angle OPM, $OP_1 = OP$. Hence, $M_1P_1 = OM$, and $OM_1 = MP$. In each figure M_1P_1, OM, have the same algebraic signs, and OM_1, MP, have opposite signs. Hence, for all angles A,

$$\sin(90° + A) = \frac{M_1P_1}{OP} = \frac{OM}{OP} = \cos A;$$

$$\cos(90° + A) = \frac{OM_1}{OP_1} = -\frac{MP}{OP} = -\sin A.$$

So also, $\tan(90° + A) = -\cot A$; $\cot(90° + A) = -\tan A$;

$\sec(90° + A) = -\operatorname{cosec} A$; $\operatorname{cosec}(90° + A) = \sec A$.

These four relations can be deduced from the figures, or by means of (1), (2), Art. 44.

D. **Ratios of $-A$.** Describe the angles A, $-A$. Let OP, OP_1, be the terminal lines of A, $-A$, respectively. In Figs. 44 *a*,

b, c, d, A is in the first, second, third, fourth quadrants, respectively. Take $OP_1 = OP$, and draw PM, P_1M_1, at right angles to the initial line. In the two triangles, OM_1P_1, OMP, (in each figure) the angles at M_1, M, are right, angle $M_1OP_1 =$ angle MOP,

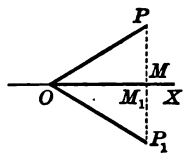

FIG. 44a.

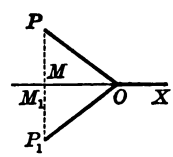

FIG. 44b.

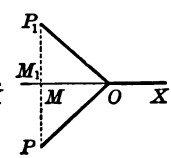

FIG. 44c.

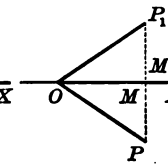

FIG. 44d.

$OP_1 = OP$. Hence, $M_1P_1 = MP$, $OM = OM_1$. In each figure, OM_1, OM, have the same sign, and M_1P_1, MP, have opposite signs. Hence, for all angles A,

$$\sin(-A) = \frac{M_1P_1}{OP_1} = -\frac{MP}{OP} = -\sin A;$$

$$\cos(-A) = \frac{OM_1}{OP_1} = \frac{OM}{OP} = \cos A.$$

So also, $\tan(-A) = -\tan A$; $\quad \cot(-A) = -\cot A$;

$$\sec(-A) = \sec A; \quad \operatorname{cosec}(-A) = -\operatorname{cosec} A.$$

The last four relations can be deduced from the figures, or by means of (1), (2), Art. 44.

Ex. 1. Show that $\sin(180° + A) = -\sin A$,

$$\cos(180° + A) = -\cos A, \ \tan(180° + A) = \tan A, \text{ etc.},$$

when A denotes an angle in any one of the four quadrants.

Ex. 2. Deduce the relations between each of the following angles and angle A, viz. $270° - A$, $270° + A$, $360° - A$, $360° + A$, $n \cdot 360° \pm A$, n being any whole number.

By means of the relations shown in this article, the ratios of any angle can be expressed in terms of the ratios of an angle between 0° *and* 45°. Thus, for example,

$$\sin 700° = \sin(360° + 340°) = \sin 340° = \sin(-20°) = -\sin 20°;$$

$$\tan 975° = \tan(2 \cdot 360° + 255°) = \tan 255° = \tan(180° + 75°) = \tan 75° = \cot 15°;$$

$$\operatorname{cosec}(-1160°) = -\operatorname{cosec} 1160° = -\operatorname{cosec}(3 \cdot 360° + 80°) = -\operatorname{cosec} 80° = -\sec 10°.$$

$$\therefore \sin 700° = -.34202; \quad \tan 975° = 3.7321;$$

$$\operatorname{cosec}(1160°) = -\sec 10° = \frac{-1}{\cos 10°} = \frac{-1}{.98481} = -1.015.$$

This property, and the property that the ratio of an angle is the co-ratio of its complement, account for the arrangement and extent of the trigonometric tables.

EXAMPLES.

1. Express the ratios of the angles in Exs. 1-7, Art. 41, in terms of ratios of angles between 0° and 45°. Also find the ratios.

2. Do likewise for the angles in Exs. 1, 3, Art. 38. Also find the ratios.

3. Do likewise for the angles in Exs. 1, 2, 3, Art. 37. Also find the ratios.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chap. V. will be found at page* 186.

CHAPTER VI.

TRIGONOMETRIC RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

N.B. *Another way of making the derivations shown in Arts. 46–48 is given in Note B of the Appendix. The method of projection, as it is called, used in Note B, is preferred by many.*

46. Derivation of the sine and cosine of the sum of two angles when each of the angles is less than a right angle. In this article and the following one, careful regard must be paid to the *directions* in which lines and angles are measured, and to the *order of the letters* used in measuring them. See Arts. 36, 37, 40.

To deduce sin $(A+B)$ and cos $(A+B)$. Let A and B be two angles each of which is less than a right angle. Let the turning line revolve from the initial line OX, and about O describe the

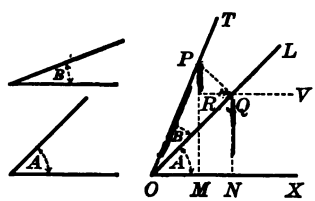

Fig. 45a.

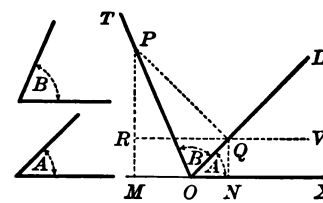

Fig. 45b.

angle XOL equal to A, and then revolve forward from the position OL and describe the angle LOT equal to B. Thus, angle $XOT = A+B$. [In Fig. 45 *a*, $A+B$ is less than 90°; in Fig. 45 *b*, $A+B$ is greater than 90°.] Take any point P on OT, the terminal line of the angle $(A+B)$, and draw PQ at right angles to OL, the terminal line of the angle A. From P, Q, draw PM, QN, at right angles to the initial line; and through Q draw VQR parallel to OX and intersecting MP in R.

$$\sin(A+B) = \sin XOP = \frac{MP}{OP} = \frac{NQ+RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP}$$

$$= \frac{OQ}{OP}\sin A + \frac{QP}{OP}\sin QP. \quad \text{(Definitions, Art. 40.)}$$

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos QOP = \cos B; \quad \frac{QP}{OP} = \sin QOP = \sin B;$$

$$\sin VQP = \sin(180° - PQR) = \sin PQR$$

$$= \cos RQO = \cos XOQ = \cos A.$$

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B. \quad (1)$$

$$\cos(A+B) = \cos XOP = \frac{OM}{OP} = \frac{ON - RQ}{OP}$$

$$= \frac{ON}{OP} + \frac{QR}{OP}$$

$$= \frac{OQ}{OP}\cos A + \frac{QP}{OP}\cos VQP.$$

(Definitions, Art. 40.)

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos B; \quad \frac{QP}{OP} = \sin B;$$

$$\cos VQP = \cos(180° - PQR) = -\cos PQR = -\sin RQO = -\sin A.$$

$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B. \quad (2)$$

EXAMPLES.

1. $\sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

2. Find $\cos 75°$ by putting $30° + 45°$ for $75°$ and using formula (2).

3. Deduce the sine and cosine of $15°$ from the results in Exs. 1, 2.

4. Find sin 90°, cos 90°, by putting $90° = 30° + 60°$. Also by putting $90° = 45° + 45°$. Also by putting $90° = 75° + 15°$.

5. Find sin 120°, cos 120°, by putting $120° = 60° + 60°$; $120° = 90° + 30°$; $120° = 75° + 45°$.

6. Find sin 150°, cos 150°, by putting $150° = 75° + 75°$; $150° = 90° + 60°$.

7. Find sin 135°, cos 135°, by putting $135° = 75° + 60°$; $135° = 90° + 45°$.

8. Given $\sin x = \frac{1}{4}$, $\sin y = \frac{1}{3}$, x and y both in the first quadrant; find $\sin(x+y)$, $\cos(x+y)$.

47. Derivation of the sine and cosine of the difference of two angles when each of the angles is less than a right angle. The construction and derivation are *very similar* to that made in the preceding article.

To deduce sin $(A-B)$ and cos $(A-B)$. Let A and B be two angles each of which is less than a right angle, and let A be the greater. Let the turning line revolve from the initial line OX,

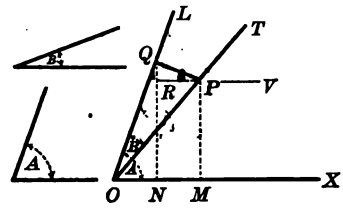

FIG. 46.

and about O describe the angle XOL equal to A, and then revolve backward from the position OL and describe the angle LOT equal to $-B$. Then angle $XOT = A - B$. Take any point P on OT, the terminal line of the angle $(A-B)$, and draw PQ at right angles to OL, the terminal line of the angle A. From P, Q, draw PM, QN, at right angles to the initial line; and through P draw RPV parallel to OX and intersecting NQ in R.

$$\sin(A-B) = \sin XOP = \frac{MP}{OP} = \frac{NQ - RQ}{OP} = \frac{NQ}{OP} - \frac{RQ}{OP}$$

$$= \frac{OQ \sin A}{OP} - \frac{PQ \sin VPQ}{OP}. \quad \text{(Definitions, Art. 40.)}$$

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos QOP = \cos(-B) = \cos B;$$

$$-\frac{PQ}{OP} = \frac{QP}{OP} = \sin QOP = \sin(-B) = -\sin B;$$

$$\sin VPQ = \sin(180° - QPR) = \sin QPR = \cos RQP = \cos A.$$

$$\therefore \sin(A - B) = \sin A \cos B - \cos A \sin B. \quad (3)$$

$$\cos(A - B) = \cos XOP = \frac{OM}{OP} = \frac{ON + RP}{OP} = \frac{ON}{OP} - \frac{PR}{OP}$$

$$= \frac{OQ \cos A}{OP} - \frac{PQ \cos VPQ}{OP}. \quad \text{(Definitions, Art. 40.)}$$

Now, by definitions in Art. 40, and by Art. 45,

$$\frac{OQ}{OP} = \cos B, \text{ and } -\frac{PQ}{OP} = -\sin B, \text{ as shown above;}$$

$$\cos VPQ = \cos(180° - QPR) = -\cos QPR = -\sin RQP = -\sin A.$$

$$\therefore \cos(A - B) = \cos A \cos B + \sin A \sin B. \quad (4)$$

If B is greater than A, then the formula,

$$\sin(B - A) = \sin B \cos A - \cos B \sin A,$$

can be deduced as above. Since

$$\sin(A - B) = -\sin(B - A),$$

then $$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

It is shown in Art. 48 *that the formulas* (1), (2), (3), (4), ***are true*** *for all values of* A *and* B. These formulas are called ***the addition*** and *subtraction formulas* or *theorems* in trigonometry. They are of such great importance, and so many thorems can be deduced by means of them, that they are called the ***fundamental*** formulas of trigonometry.* They should be memorized.

Note. Arts. 48, 49, may be omitted, if deemed advisable, until after the solution of triangles is completed. Art. 48 can also be shown ***geometrically.***

---

* Adrian Romanus (1561–1625), professor of mathematics and medicine at the University of Louvain, was the first to prove the formula for $\sin(A + B)$. The formulas for $\cos(A \pm B)$ and $\sin(A - B)$ were given by Pitiscus (1561–1613), a German mathematician and astronomer, in his Trigonometry published in 1595.

EXAMPLES.

1. Derive sin 15°, cos 15°, on putting 60° − 45° for 15°.
2. Derive sin 15°, cos 15°, on putting 45° − 30° for 15°.
3. Find sin $(x-y)$, cos $(x-y)$ in the cases in Ex. 8, Art. 46.

48. Proof of addition and subtraction formulas for all values of A and B. These formulas have been proved in Arts. 46, 47; for values of A and B which are less than a right angle. In Art. 45 c it has been shown that for *any* angle, say X,

$$\cos X = \sin(90° + X), \quad \sin X = -\cos(90° + X).$$

Hence, $\cos A = \sin(90° + A)$, $\sin A = -\cos(90° + A)$,

$\cos(A+B) = \sin(90° + A + B)$, $\sin(A+B) = -\cos(90° + A + B)$.

The substitution of these values for cos A, sin A, cos $(A+B)$, sin $(A+B)$, in (1), Art. 46, gives

$$-\cos(90°+A+B) = -\cos(90°+A)\cos B + \sin(90°+A)\sin B;$$

$$\therefore \cos(\overline{90°+A}+B) = \cos(90°+A)\cos B - \sin(90°+A)\sin B. \quad (1)$$

The substitution of the same values in (2), Art. 46, gives

$$\sin(\overline{90°+A}+B) = \sin(90°+A)\cos B + \cos(90°+A)\sin B. \quad (2)$$

Hence, formulas (1), (2), Art. 46, are true when one of the angles is increased by a right angle. In a similar way, these formulas can be shown to remain true when one of the angles in (1), (2), of this article is increased by a right angle. It is thus evident that the formulas are true, no matter how many right angles are added to either one or both of the angles. It can easily be shown that $\sin A = \cos(A-90°)$, $\cos A = -\sin(A-90°)$. Then, in the same way as that just employed, it can be shown that the formulas (1), (2), Art. 46, hold when either one or both of the angles is diminished by integral multiples of 90°. Hence, formulas (1), (2), Art. 46, are true for angles in any quadrant, that is, for all angles. In a similar way, formulas (3), (4), Art. 47, can be shown to be universally true.

49. Each fundamental formula contains the others. From any one of the four fundamental formulas, the remaining three can be derived. Thus for example:

In (1) Art. 46, change A into $90° - A$; then

$$\sin(90° - A + B) = \sin(90° - A)\cos B + \cos(90° - A)\sin B.$$

From this,

$$\sin(90° - \overline{A - B}), \textit{ i.e. } \cos(A - B) = \cos A \cos B + \sin A \sin B.$$

In (1) Art. 46, change B into $(-B)$; then

$$\begin{aligned}\sin(A - B) &= \sin A \cos(-B) + \cos A \sin(-B)\\ &= \sin A \cos B - \cos A \sin B.\end{aligned}$$

In (1) Art. 46, change A into $(90° + A)$; then

$$\sin(90° + A + B) = \sin(90° + A)\cos B + \cos(90° + A)\sin B,$$

whence, $\cos(A + B) = \cos A \cos B - \sin A \sin B.$

Ex. 1. From formula (2), Art. 46, derive the other three fundamental formulas.

Ex. 2. So also, from formula (3), Art. 47.

Ex. 3. So also, from formula (4), Art. 47.

50. Ratio of an angle in terms of the ratios of its half angle. In this article and Arts. 51, 52, a few deductions will be made from the addition and subtraction formulas, which have been shown to be true for all angles. These deductions are necessary for the explanations concerning triangles, as well as useful for other purposes. More ample opportunity will be afforded later for working exercises involving the use of these formulas. The fundamental formulas may be brought together:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B. \quad (1)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B. \quad (2)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B. \quad (3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \quad (4)$$

Let $B = A$; then, from (1),

$$\sin(A + A) = \sin A \cos A + \cos A \sin A;$$

that is, $$\sin 2A = 2 \sin A \cos A. \quad (5)$$

Similarly, from (3), $$\cos 2A = \cos^2 A - \sin^2 A. \quad (6)$$

Since $\cos^2 A + \sin^2 A = 1$, it follows that

$$\cos 2A = 1 - 2 \sin^2 A. \quad (7)$$

and $$\cos 2A = 2 \cos^2 A - 1. \quad (8)$$

In formulas (1)–(8), A, B, denote any angles whatsoever. These formulas occur so often, and are so useful, that it is well to translate them into words. Thus,

A

sine *sum* of any two angles = sin *first* · cosine *second* + cosine *first* · sine *second*

sine *difference* of any two angles = sine *first* · cosine *second* − cosine *first* · sine *second*

cosine *sum* of any two angles = cosine *first* · cosine *second* − sine *first* · sine *second*

cosine *difference* of any two angles = cosine *first* · cosine *second* + sine *first* · sine *second*

Since A is one-half of $2A$, formulas (5)–(8) can be translated as follows:

sine any *angle* = 2 sine *half-angle* · cosine *half-angle*,

cosine any *angle* = (cosine *half-angle*)² − (sine *half-angle*)²,
= 1 − 2 (sine *half-angle*)²,
= 2 (cosine *half-angle*)² − 1.

EXAMPLES.

1. Find $\cos 22\frac{1}{2}°$ from $\cos 45°$.

$2 \cos^2 22\frac{1}{2}° = 1 + \cos 45°$ by (8);

$\therefore \cos^2 22\frac{1}{2}° = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{1 + \sqrt{2}}{2\sqrt{2}} = \frac{1 + 1.4142}{2 \times 1.4142} = .8536;$

$\therefore \cos 22\frac{1}{2}° = .9239.$

2. Express $\cos 4x$ in terms of $\sin x$ and $\cos x$.

$\cos 4x = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1 = 8\cos^4 x - 8\cos^2 x + 1$

3. Deduce $\sin 30°$, $\cos 30°$, from $\cos 60°$.

4. Deduce $\sin 75°$, $\cos 75°$, from $\cos 150°$. [Logarithms may be helpful.]

5. Deduce $\sin 22\frac{1}{2}°$, from $\cos 45°$.

6. Deduce $\sin 15°$, $\cos 15°$, from $\cos 30°$.

7. Express $\cos 6x$, $\sin 6x$, in terms of ratios of $3x$.

8. Express $\cos 3x$, $\sin 3x$, in terms of ratios of $\frac{1}{3}x$.

9. Express $\sin \frac{1}{4}x$, $\cos \frac{1}{4}x$, in terms of ratios of $\frac{1}{8}x$.

10. Express $\cos 6x$, $\sin 6x$, in terms of ratios of $12x$.

11. Express $\cos 3x$, $\sin 3x$, in terms of ratios of $6x$.

12. Express $\sin \frac{3}{4}x$, $\cos \frac{3}{4}x$, in terms of ratios of $\frac{3}{8}x$.

13. Show that $\sin(n+1)A + \sin(n-1)A = 2\sin nA\cos A$, and

$\cos(n+1)A + \cos(n-1)A = 2\cos nA\cos A$; and

hence, express $\sin 2A$, $\cos 2A$, in terms of $\sin A$, $\cos A$.

51. Tangents of the sum, and difference of two angles, and of twice an angle. Let A, B, be any two angles. It is required to find $\tan(A+B)$ and $\tan(A-B)$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

On dividing each term of the numerator and the denominator of the second member by $\cos A\cos B$, there is obtained

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}. \qquad (1)$$

In the same way it can be shown that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}. \qquad (2)$$

Formula (2) can also be deduced from (1) by changing B into $-B$.

If $B = A$, then (1) becomes

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \qquad (3)$$

CHAPTER VII.

SOLUTION OF TRIANGLES IN GENERAL.

53. Cases for solution. In Art. 34 oblique triangles were solved by means of right-angled triangles. In this chapter some relations of the sides and angles of any triangle (whether right-angled or oblique) will be derived; methods of solution will be shown, which are applicable to the solution of both right-angled and oblique triangles, and which are independent of the special aid that can be afforded by right-angled triangles. In Art. 54 the chief relations between the sides and angles of a triangle will be deduced. These relations constitute the foundation for the remainder of the chapter. In Arts. 55–58 solutions of triangles are obtained without the use of logarithms; in Arts. 60–62 logarithms are employed in finding the solutions.

In order that a triangle may be constructed, three elements, one of which must be a side, are required. Hence, there are four cases for construction and solution, namely, when the given parts are as follows:

I. **One side** and **two angles.**

II. **Two sides** and **the angle opposite to one of them.**

III. **Two sides** and **their included angle.**

IV. **Three sides.**

Before proceeding, the student should test his ability to construct a triangle readily in each of these cases.

In the discussions that follow, the triangle is denoted by ABC, the angles by A, B, C, and the lengths of their opposite sides by a, b, c, respectively.*

---

* The formulas are greatly simplified by the adoption of this notation, which was first introduced by Leonhard Euler (1707–1783).

54. Fundamental relations between the sides and angles of a triangle. The law of sines. The law of cosines.

I. *The law of sines.* From C in the triangle ABC draw CD at right angles to opposite side AB, and meeting AB or AB produced in D. (In Fig. 47 *a* B is acute, in Fig. 47 *b* B is obtuse, and in

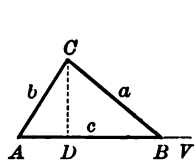

FIG. 47a.

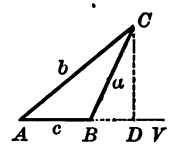

FIG. 47b.

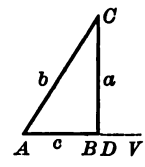

FIG. 47c.

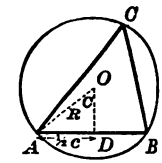

FIG. 48.

Fig. 47 *c* B is a right angle.) Produce AB to V. In what follows, AB is taken as the positive direction.

In CDA, $$DC = b \sin A.$$

In CDB (Figs. 47 *a*, *b*), $DC = a \sin VBC$ (Definition, Art. 40.)

$$= a \sin B.$$

[$\because \sin VBC = \sin(180° - VBC)$, Art. 45, $= \sin CBA$]

In Fig. 47 *c*, $$DC = BC = a = a \sin B.$$

($\because B = 90°$, and $\sin 90° = 1$)

Therefore, in all three triangles,

$$a \sin B = b \sin A.$$

Hence, $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, on drawing a line from B at right angles to AC, it can be shown that

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Hence, in any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (1)$$

In words: *The sides of any triangle are proportional to the sines of the opposite angles.*

Each of the fractions in (1) gives *the length of the diameter of the circle described about ABC.* Let O (Fig. 48) be the centre and R the radius of the circle described about ABC. Draw OD at right angles to any one of the sides, say AB. Draw AO. Then $AD = \frac{1}{2}c$, $AOD = C$, by geometry. In triangle AOD,

$$AD = AO \sin AOD; \text{ i.e. } \tfrac{1}{2}c = R \sin C.$$

$$\therefore 2R = \frac{c}{\sin C}. \qquad (2)$$

Ex. 1. Explain why the circumscribing circle of a triangle depends only upon *one side and its opposite angle.*

Ex. 2. Derive the law of sines by drawing a perpendicular from A to BC.

Ex. 3. Derive $2R = \frac{a}{\sin A}$, $2R = \frac{b}{\sin B}$, by means of figures.

II. *The law of cosines.* An expression for the length of the side of a triangle in terms of the cosine of the opposite angle and the lengths of the other two sides, will now be deduced. The angle A is acute in Fig. 49 a, obtuse in Fig. 49 b, right in Fig. 49 c. From C draw CD at right angles to AB. The direction AB is taken as positive.

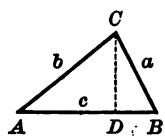

FIG. 49a.

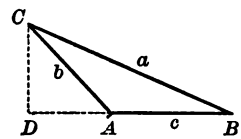

FIG. 49b.

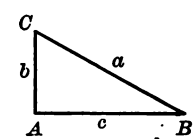

FIG. 49c.

In Figs. 49 a, 49 b, $\overline{BC}^2 = \overline{DC}^2 + \overline{DB}^2$.

In Fig. 49 a, $DB = AB - AD$;

in Fig. 49 b, $DB = DA + AB = -AD + AB$.

Hence, in both figures, $\overline{BC}^2 = \overline{DC}^2 + (AB - AD)^2$

$= \overline{DC}^2 + \overline{AD}^2 + \overline{AB}^2 - 2\,AB \cdot AD$.

650044

In Fig. 49 *a*, $AD = AC \cos BAC$;

in Fig. 49 *b*, $AD = AC \cos BAC$ (Art. 40).

Also, $\overline{DC}^2 + \overline{AD}^2 = \overline{AC}^2$.

Hence, in both figures, $\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AC \cdot AB \cos A$;

that is, $$a^2 = b^2 + c^2 - 2\,bc \cos A. \qquad (3)$$

This formula also holds for Fig. 49 *c*; for there,

$$\cos A = \cos 90° = 0.$$

Similar formulas for *b*, *c*, can be derived in like manner, or can be obtained from (3) by symmetry:

$$b^2 = c^2 + a^2 - 2\,ca \cos B, \quad c^2 = a^2 + b^2 - 2\,ab \cos C. \qquad (3')$$

These formulas can be expressed in words: *In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides multiplied by the cosine of their included angle.*

Note. In Fig. 49 *a*, A is acute and $\cos A$ is positive; in Fig. 49 *b*, A is obtuse and $\cos A$ is negative. Hence formula (3) shows that in Fig. 49 *a*, a^2 is less than $b^2 + c^2$, and that in Fig. 49 *b*, a^2 is greater than $b^2 + c^2$. In Fig. 49 *c*, $a^2 = b^2 + c^2$.

Relation (3) may be expressed as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \qquad (4)$$

and similarly for $\cos B$, $\cos C$.

Ex. Derive the formulas for b^2 and for c^2.

Each of the relations (1), (3), (3'), involves *four* of the six elements of a triangle. If any *three* of the elements in any one of these relations are known, then the *fourth* element can be found by solving the equation. Inspection shows that relations (1) are serviceable in the solution of Cases I., II., Art. 53, and that relations (3), (3'), are serviceable in the solution of Cases III., IV., Art. 53. The student is advised to try to work some of the examples in Arts. 55–58 before reading the text of the articles. (See Arts. 20–24, 34.)

54 a. Substitution of sines for sides, and of sides for sines.

Since $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, the sines of the opposite angles can be substituted for the sides of triangles, and *vice versa*, when they are involved *homogeneously* in the numerator and denominator of a fraction, or in both members of an equation.

Thus, on putting $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=x$, it follows that $a=x\sin A$, $b=x\sin B$, $c=x\sin C$.

Then, for example, $\frac{a^2}{b+c}=\frac{ax\sin A}{x\sin B+x\sin C}=\frac{a\sin A}{\sin B+\sin C}$.

EXAMPLES.

1. Show that in any triangle $\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$.

$$\frac{a+b}{c}=\frac{\sin A+\sin B}{\sin C}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}C\cos\frac{1}{2}C}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C},$$

for $\sin\frac{1}{2}(A+B)=\cos\frac{1}{2}C$, since $\frac{1}{2}(A+B)+\frac{1}{2}C=90°$.

2. Derive two other relations similar to that in Ex. 1.

3. Show that

$$\frac{3a^2+2b^2}{abc}=\frac{3\sin^2A+2\sin^2B}{a\sin B\sin C}=\frac{3\sin^2A+2\sin^2B}{b\sin A\sin C}=\frac{3\sin^2A+2\sin^2B}{c\sin A\sin B}.$$

55. Case I. Given one side and two angles. In triangle ABC, suppose that A, B, a are known; it is required to find C, b, c. In this case (see Fig. 47 *a*, Art. 54),

$$C=180°-(A+B);$$

$$\frac{b}{\sin B}=\frac{a}{\sin A},\text{ whence }b=\frac{a}{\sin A}\cdot\sin B;$$

$$\frac{c}{\sin C}=\frac{a}{\sin A},\text{ whence }c=\frac{a}{\sin A}\cdot\sin C.$$

Checks: $a^2=b^2+c^2-2bc\cos A$, $\frac{b}{\sin B}=\frac{c}{\sin C}$, the result in Ex. 1, Art. 54 *a*. Other checks will be discovered later.

EXAMPLES.

1. Solve the triangle PQR, given: $PQ = 12$ in., $Q = 40°$, $P = 75°$. Solution: $R =$ $PR =$ $RQ =$

$R = 180° - (P + Q) = 180° - (40° + 75°) = 65°$.

$$\frac{PR}{\sin Q} = \frac{PQ}{\sin R}. \qquad \frac{RQ}{\sin P} = \frac{PQ}{\sin R}.$$

$$\therefore PR = \frac{PQ}{\sin R} \cdot \sin Q, \qquad RQ = \frac{PQ}{\sin R} \cdot \sin P,$$

$$= \frac{12}{\sin 65°} \cdot \sin 40°, \qquad = \frac{12}{\sin 65°} \cdot \sin 75°$$

$$= \frac{12}{.9063} \times .6428, \qquad = \frac{12}{.9063} \times .9659,$$

$$= 13.24 \times .6428, \qquad = 13.24 \times .9659,$$

$$= 8.51 \text{ in.} \qquad = 12.8 \text{ in.}$$

2. In ABC, $A = 50°$, $B = 75°$, $c = 60$ in. Solve the triangle.

3. In ABC, $A = 131° 35'$, $B = 30°$, $b = 5\frac{1}{3}$ ft. Find a.

4. In ABC, $B = 70° 30'$, $C = 78° 10'$, $a = 102$. Solve the triangle.

5. In ABC, $B = 98° 22'$, $C = 41° 1'$, $a = 5.42$. Solve the triangle.

56. Case II. Given two sides and an angle opposite to one of them. In the triangle ABC let a, b, A be known, and C, B, c be required. The triangle will first be constructed with the given

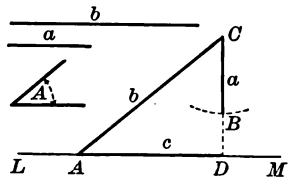

FIG. 50.

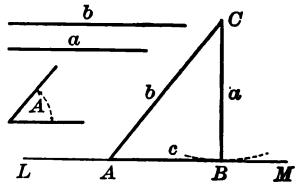

FIG. 51.

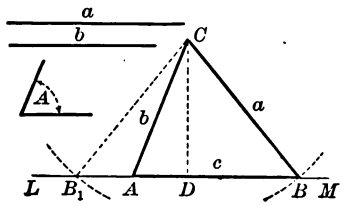

FIG. 52.

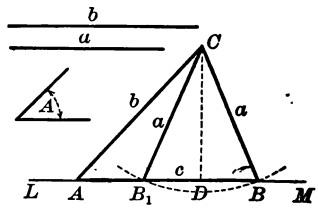

FIG. 53.

elements. At any point A of a straight line LM, unlimited in length, make angle MAC equal to angle A, and cut off AC equal to b. About C as a centre, and with a radius equal to a, describe a circle. This circle will either:

(1) Not reach to LM, as in Fig. 50.

(2) Just reach to LM, thus having LM for a tangent, as in Fig. 51.

(3) Intersect LM in two points, as in Figs. 52, 53.

Each of these possible cases must be considered. In each figure, from C draw CD at right angles to AM; then $CD = b \sin A$.

In case (1), Fig. 50, $CB < CD$, and there is no triangle which can have the given elements. Hence, *the triangle is impossible when* $a < b \sin A$.

In case (2), Fig. 51, $CB = CD$. Hence, *the triangle which has elements equal to the given elements is right-angled when* $a = b \sin A$.

In case (3), Figs. 52, 53, $CB > CD$; that is, $a > b \sin A$. If $a > b$, then the points B, B_1, in which the circle intersects LM, are on opposite sides of A, as in Fig. 52, and *there is one triangle which has three elements equal to the given elements*, namely, ABC. If $a < b$, then the points of intersection B, B_1, are on the same side of A, as in Fig. 53, and *there are two triangles which have elements equal to the given elements*, namely, ABC, AB_1C. For, in ABC, angle $BAC = A$, $AC = b$, $BC = a$; in AB_1C, angle $B_1AC = A$, $AC = b$, $B_1C = a$. *Both triangles must be solved.* In this case, Fig. 53, the given angle is opposite to the smaller of the two given sides. Hence, *there may be two solutions when the given angle is opposite to the smaller of the two given sides.* The words "*may be*" are used, for in cases (1), (2), the given angle is opposite to the smaller of the two given sides. Case II. is sometimes called **the ambiguous case** in the solution of triangles.

The ambiguity in Case II. is also apparent in the trigonometric solution. The angle B is found by means of the relation,

$$\frac{\sin B}{b} = \frac{\sin A}{a}; \text{ or, } \sin B = \frac{b}{a} \sin A. \qquad (1)$$

The angle B is thus determined from its sine. Now there is always an *ambiguity* when an angle of a triangle is determined

from its sine alone, for $\sin x = \sin(180° - x)$. Figure 53 shows the two angles which have the same sine, namely, ABC, AB_1C. In Fig. 52, the given condition, namely, that $b < a$, shows that $B < A$; accordingly, only the acute angle corresponding to $\sin B$ can be taken. If, in equation (1), $b \sin A > a$, then $\sin B > 1$, and, accordingly, B is impossible and there is no solution. If, in equation (1), $b \sin A = a$, then $\sin B = 1$, and $B = 90°$. The consideration of the trigonometric equation (1) leads, therefore, to the same results as the preceding geometrical investigation.

Checks: $A + B + C = 180°$, and, as in Case I. Other checks will be found later.

EXAMPLES.

1. Solve the triangle STV, given: $ST = 15$, $VT = 12$, $S = 52°$.

$$\frac{\sin V}{ST} = \frac{\sin S}{VT};$$

$$\frac{\sin V}{15} = \frac{\sin 52°}{12} = \frac{.78801}{12} = .065668.$$

$$\therefore \sin V = 15 \times .065668 = .98502.$$

$\therefore V = 80° 4' 20''$, or $180° - 80° 4' 20''$, *i.e.* $99° 55' 40''$.

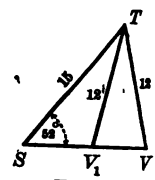

FIG. 54.

Both values of V must be taken, since the given angle is opposite to the smaller of the given sides. The two triangles corresponding to the two values of V are STV, STV_1, Fig. 54, in which

$$SVT = 80° 4' 20'', \quad SV_1T = 99° 55' 40''.$$

In STV_1

angle $STV_1 = 180° - (S + SV_1T)$
$= 28° 4' 20''$.

$$\frac{SV_1}{\sin STV_1} = \frac{V_1T}{\sin S};$$

$$\frac{SV_1}{.47059} = \frac{12}{.78801} = 15.228.$$

$$\therefore SV_1 = 7.17.$$

In STV

angle $STV = 180° - (S + SVT)$
$= 47° 55' 40''$.

$$\frac{SV}{\sin STV} = \frac{VT}{\sin S};$$

$$\frac{SV}{.74230} = 15.228.$$

$$\therefore SV = 11.3.$$

The solutions are:

$$\left.\begin{aligned} V_1 &= 99° 55' 40'' \\ T &= 28° 4' 20'' \\ SV_1 &= 7.17 \end{aligned}\right\}; \qquad \left.\begin{aligned} V &= 80° 4' 20'' \\ T &= 47° 55' 40'' \\ SV &= 11.3 \end{aligned}\right\}.$$

In the ambiguous case, care must be taken that the calculated sides and angles are combined properly.

2. Solve ABC, given: $a = 29$ ft., $b = 34$ ft., $A = 30° 20'$.
3. Solve ABC when $a = 30$ ft., $b = 24$ ft., $B = 65°$.
4. Solve ABC when $a = 30$ in., $b = 24$ in., $A = 65°$.
5. Solve ABC when $a = 15$ ft., $b = 8$ ft., $B = 23° 25'$.

57. Case III. Given two sides and their included angle. In the triangle ABC, a, b, C are known, and it is required to find A, B, c. In this case, c can be determined from the relation $c^2 = a^2 + b^2 - 2ab\cos C$, Art. 54; angle A can be determined from the relation

$$\frac{\sin A}{a} = \frac{\sin C}{c};$$

angle B can be determined from the relation

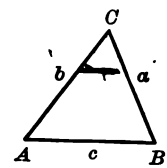

FIG. 55.

$$A + B + C = 180°, \text{ or from } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Checks: $a^2 = b^2 + c^2 - 2bc\cos A$, $b^2 = a^2 + c^2 - 2ac\cos A$, the result in Ex. 1, Art. 54 *a*; other checks will be found later.

EXAMPLES.

1. In triangle PQR, $p = 8$ ft., $r = 10$ ft., $Q = 47°$. Find q, P, R.

$$q^2 = p^2 + r^2 - 2pr\cos Q$$
$$= 64 + 100 - 2 \times 8 \times 10 \times .6820 = 54.88.$$
$$\therefore\ q = \sqrt{7.408}.$$
$$\sin P = \frac{p\sin Q}{q} = \frac{8 \times .7314}{7.408} = .7898. \quad \therefore\ P = 52° 10'.$$
$$\sin R = \frac{r\sin Q}{q} = \frac{10 \times .7314}{7.408} = .9873. \quad \therefore\ R = 80° 50'.$$

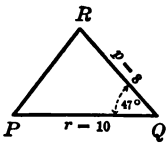

FIG. 56.

2. Solve ABC, given: $a = 34$ ft., $b = 24$ ft., $C = 59° 17'$.
3. Solve ABC, given: $a = 33$ ft., $c = 30$ ft., $B = 35° 25'$.
4. Solve RST, given: $r = 30$ ft., $s = 54$ ft., $T = 46°$.
5. Solve PQR, given: $p = 10$ in., $q = 16$ in., $R = 97° 54'$.

58. Case IV. Three sides given. If the sides a, b, c are known in the triangle ABC, then the angles A, B, C can be found by means of the relations (3), Art. 54.

Checks: Relations (1), Art. 54: $A + B + C = 180°$. Other checks will be shown later.

EXAMPLES.

1. In ABC, $a = 4$, $b = 7$, $c = 10$; find A, B, C.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{49 + 100 - 16}{2 \times 7 \times 10} = \frac{133}{140} = .9500. \qquad \therefore A = 18° 12'.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{100 + 16 - 49}{2 \times 10 \times 4} = \frac{67}{80} = .8375. \qquad \therefore B = 33° 7' 30''.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 49 - 100}{2 \times 4 \times 7} = \frac{-35}{56} = -.6250. \qquad \therefore C = 128° 40' 52''.$$

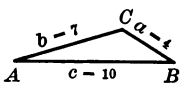

FIG. 57.

Angle C is in the *second* quadrant since its cosine is negative.

Check: $18° 12' + 33° 7' 30'' + 128° 40' 52'' = 180° 0' 22''$. The discrepancy is due to the fact that four-place tables were used in the computation. Had five-place tables been used, the discrepancy would have been less.

2. In PQR, $p = 9$, $q = 24$, $r = 27$. Find P, Q, R.
3. In RST, $r = 21$, $s = 24$, $t = 27$. Find R, S, T.
4. In ABC, $a = 12$, $b = 20$, $c = 28$. Find A, B, C.
5. In ABC, $a = 80$, $b = 26$, $c = 74$. Find A, B, C.
6. Solve Ex. 1, using five-place tables.

59. The aid of logarithms in the solution of triangles. It was pointed out in Art. 6 that an expression is adapted for logarithmic computation when, and only when, it is decomposed into factors. In Cases I., II., Arts. 55, 56, the expressions used in solving the triangle can be computed with the help of logarithms. On the other hand, the side opposite to the given angle in Case III., Art. 57, and the angles in Case IV., Art. 58, are found by evaluating expressions which are not adapted to the use of logarithms. Other relations between the sides and angles of a triangle will be found in Arts. 61, 62. By these relations the computations in Cases III., IV., can be made both without and with logarithms. *These relations are useful not merely for purposes of computation; they are important in themselves, and valuable because many important properties of triangles can be deduced from them.*

The explanations given in Arts. 55–57 are presupposed in Arts. 60–62. The general directions to be observed in working the problems are as follows:

1. *Write down all the formulas which will be used in the computation.*

2. *Express these formulas in the logarithmic form.*

[As soon as the student perceives that this step does not afford any additional assistance, it may be omitted. See Art. 27, Ex. 1, Note 6.]

3. *Make a skeleton scheme, and arrange the arithmetical work neatly and clearly.*

The skeleton schemes in the worked examples that follow, are apparent when the numbers are omitted.*

Checks: The various formulas can serve as checks on the results of one another. The relations derived in Exs. 1, 2, Art. 54 *a*, are also useful as checks.

60. The use of logarithms in Cases I., II. An example worked out, will give sufficient explanation.

EXAMPLES.

1. In ABC, given: $a = 447$, $b = 576$, $A = 47° 35'$. To find: $B =$, $C =$, $c =$ (Write the results here.)

Since $a < b$, there may be two solutions. Construction shows there are two solutions.

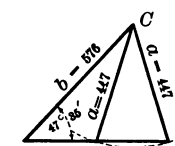

FIG. 58.

Formulas: $\sin ABC = \frac{b}{a} \sin A = \sin AB_1C.$

$ACB = 180° - (A + ABC)$. $ACB_1 = 180° - (A + AB_1C)$.

$AB = \frac{a}{\sin A} \sin ACB.$ $AB_1 = \frac{a}{\sin A} \sin ACB_1.$

$$\therefore \log\sin ABC = \log b + \log\sin A - \log a = \log\sin AB_1C;$$

$$\log AB = \log a + \log\sin ACB - \log\sin A;$$

$$\log AB_1 = \log a + \log\sin ACB_1 - \log\sin A.$$

---

* Cologarithms are not used in the solutions in the text. In extensive computations the use of cologarithms is favoured by many computers; but it seems best for *beginners* in trigonometry first to become accustomed to the obvious and direct method of working with logarithms.

$$\log a = 2.65031$$
$$\log b = 2.76042$$
$$\log \sin A = 9.86821 - 10$$
$$\therefore \log \sin B = 9.97832 - 10$$

$\therefore ABC = 72° 2' 45''$
$\therefore ACB = 60° 22' 15''$
$\log \sin ACB = 9.93914 - 10$
$\therefore \log AB = 2.72124$
$\therefore AB = 526.3$

and $AB_1C = 107° 57' 15''$
$\therefore ACB_1 = 24° 27' 45''$
$\log \sin ACB_1 = 9.61710 - 10$
$\therefore \log AB_1 = 2.39920$
$\therefore AB_1 = 250.7$

In obtaining $\log AB$, for instance, $\log \sin ACB$ may be written on the margin of a slip of paper, placed under $\log a$, the addition made, $\log \sin A$ placed beneath, and the subtraction made.

Solve the triangle ABC, when the following elements are given:

2. $A = 63° 48'$, $B = 49° 25'$, $a = 825$ ft.
3. $B = 128° 3' 49''$, $C = 33° 34' 47''$, $a = 240$ ft.
4. $A = 78° 30'$, $b = 137$ ft., $a = 65$ ft.
5. $a = 275.48$, $b = 350.55$, $B = 60° 0' 32''$.
6. $c = 690$, $a = 464$, $A = 37° 20'$.
7. $a = 690$, $b = 1390$, $A = 21° 14' 25''$.

61. Relation between the sum and difference of any two sides of a triangle. The Law of Tangents. Use of logarithms in Case III. In any triangle ABC, for any two sides, say a, b,

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$ [By equation (1), Art. 54.]

$$\therefore \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$ [By composition and division.]

$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}.$$ [Art. 52, Formulas (5), (6).]

$$\therefore \frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$ [Art. 44, A, B.] (1)

That is, *the difference of any two sides of a triangle is to their sum as the tangent of half the difference of their opposite angles is*

to the tangent of half their sum. This is sometimes called the **law of tangents.**

Now $A+B=180°-C$, and, consequently, $\frac{1}{2}(A+B)=90°-\frac{C}{2}$. Hence, $\tan\frac{1}{2}(A+B)=\cot\frac{C}{2}$, and, accordingly, relation (1) may be written

$$\tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\tfrac{1}{2}C. \qquad (2)$$

Formulas for b, c, and c, a, similar to the formulas for a, b in (1), (2), can be derived in the same way as (1), (2), have been derived. These formulas can also be written down immediately, on noticing the symmetry in formulas (1), (2).

Ex. *Write* the formulas for sides b, c and c, a. Derive these formulas.

Case III. In a triangle ABC, a, b, C, are known, and c, B, A, are required. Here, $\frac{1}{2}(A+B)=90°-\frac{1}{2}C$; also, $\frac{1}{2}(A-B)$ can be found by (2). Hence, A and B can be found; for

$$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B), \text{ and } B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B).$$

The side c can then be found by (1), Art. 54. (In using (1), (2), write the greater side and the greater angle first, in order that the difference may be positive.) *Formulas* (1), (2), *can also be used as a check in the cases discussed in the preceding articles.* Other checks will be shown in the next article.

EXAMPLES.

1. In triangle ABC, Given: $b=472$, $c=324$, $A=78°\,40'$. Find: $B=$, $C=$, $a=$.

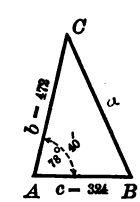

FIG. 59.

Formulas: $\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{1}{2}A.$

$\frac{1}{2}(B+C)=90°-\frac{1}{2}A.$

$B=\frac{1}{2}(B+C)+\frac{1}{2}(B-C).$

$C=\frac{1}{2}(B+C)-\frac{1}{2}(B-C).$

$a=\frac{b\sin A}{\sin B}$; or $=\frac{c\sin A}{\sin C}$.

Checks: $A+B+C=180°$, formulas in preceding articles, and formulas shown in the next article.

$\log \tan \frac{1}{2}(B - C) = \log(b - c) + \log \cot \frac{1}{2} A - \log(b + c)$,

$\log a = \log b + \log \sin A - \log \sin B$; or $= \log c + \log \sin A - \log \sin C$.

$b = 472$	$\log(b - c) = 2.17026$	$\log b = 2.67394$
$c = 324$	$\log(b + c) = 2.90091$	$\log \sin A = 9.99145 - 10$
$A = 78° 40'$	$\log \cot \frac{1}{2} A = 10.08647 - 10$	$\log \sin B = 9.95223 - 10$
$b - c = 148$	$\therefore \log \tan \frac{1}{2}(B - C) = 9.35582 - 10$	$\therefore \log a = 2.71316$
$b + c = 796$	$\therefore \frac{1}{2}(B - C) = 12° 47' 1''$	$\therefore a = 516.6$
$\frac{1}{2} A = 39° 20'$	$\frac{1}{2}(B + C) = 50° 40'$	
	$\therefore B = 63° 27' 1''$	
	$\therefore C = 37° 52' 59''$	

Check: $A + B + C = 78° 40' + 63° 27' 1'' + 37° 52' 59'' = 180°$.

Note. Formulas (1), (2), are adapted to logarithmic computation; but the computations can be made without the aid of logarithms.

2. Solve ABC, given $b = 352$, $a = 266$, $C = 73°$.

3. Solve PQR, given $p = 91.7$, $q = 31.2$, $R = 33° 7' 9''$.

4. Solve ABC, given $a = 960$, $b = 720$, $C = 25° 40'$.

5. Solve ABC, given $b = 9.081$, $c = 3.6545$, $A = 68° 14' 24''$.

6. Solve Exs. 1, 5, Art. 57, using the formulas of this article, without logarithms.

62. Trigonometric ratios of the half angles of a triangle. Use of logarithms in Case IV. In any triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \qquad \text{[Art. 54 (4).]}$$

Now $$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A,$$

and $$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A. \qquad \text{[Art. 50 (7), (8).]}$$

Also, $$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - (b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{[a - (b - c)][a + (b - c)]}{2bc}.$$

$$\therefore 2 \sin^2 \tfrac{1}{2} A = \frac{(a - b + c)(a + b - c)}{2bc} \qquad (1)$$

$$\text{Also, } 1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc} = \frac{2bc+(b^2+c^2-a^2)}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc}.$$

$$\therefore\ 2\cos^2\tfrac{1}{2}A = \frac{(b+c+a)(b+c-a)}{2bc}. \tag{2}$$

Let $a+b+c=2s$;

then $2(s-c) = (a+b+c)-2c = a+b-c$

Similarly $2(s-b) = a-b+c,$

$2(s-a) = -a+b+c.$

The substitution of these values in (1) and (2) gives

$$2\sin^2\tfrac{1}{2}A = \frac{2(s-b)\cdot 2(s-c)}{2bc};\quad 2\cos^2\tfrac{1}{2}A = \frac{2s\cdot 2(s-a)}{2bc}.$$

$$\therefore\ \sin^2\tfrac{1}{2}A = \frac{(s-b)(s-c)}{bc}; \tag{3}$$

$$\cos^2\tfrac{1}{2}A = \frac{s(s-a)}{bc}. \tag{4}$$

Since $\tan^2\frac{1}{2}A = \sin^2\frac{1}{2}A \div \cos^2\frac{1}{2}A$, it follows that

$$\tan^2\tfrac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)}. \tag{5}$$

$$\therefore\ \sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}};\quad \cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}};$$

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

NOTE. By geometry, $b+c>a$. Hence, $-a+b+c>0$, and, accordingly, $s-a$ is positive. Similarly, $s-b$, $s-c$, are also positive. Therefore, the quantities under the radical signs are positive. The positive sign must be given to the radical, for A is less than 180°, and consequently $\frac{1}{2}A$ lies between 0° and 90°.

Similar formulas hold for $\frac{1}{2}B$ and $\frac{1}{2}C$. They can be deduced in the same manner as those for $\frac{1}{2}A$; or, they can be written immediately, from the symmetry apparent in the formulas (3)–(5). The student is advised to derive the similar formulas for $\frac{1}{2}B$, $\frac{1}{2}C$, viz.:

$$\sin^2\tfrac{1}{2}B = \frac{(s-a)(s-c)}{ac}; \qquad \cos^2\tfrac{1}{2}B = \frac{s(s-b)}{ac}; \tag{3'}$$

$$\sin^2\tfrac{1}{2}C = \frac{(s-a)(s-b)}{ab}. \qquad \cos^2\tfrac{1}{2}C = \frac{s(s-c)}{ab}. \tag{4'}$$

$$\tan^2\tfrac{1}{2}B = \frac{(s-a)(s-c)}{s(s-b)}; \qquad \tan^2\tfrac{1}{2}C = \frac{(s-a)(s-b)}{s(s-c)}. \tag{5'}$$

Formula (5) can be given a more symmetrical form; for, on multiplying and dividing its second member by $(s-a)$,

$$\tan^2\tfrac{1}{2}A = \frac{(s-a)(s-b)(s-c)}{s(s-a)^2};$$

whence $$\tan\tfrac{1}{2}A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{6}$$

If $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{7}$$

then $$\tan\tfrac{1}{2}A = \frac{r}{s-a}. \tag{8}$$

Similarly, $$\tan\tfrac{1}{2}B = \frac{r}{s-b}, \quad \tan\tfrac{1}{2}C = \frac{r}{s-c}. \tag{8'}$$

When all the sides are known, the angles can be found by means of formulas (3)–(5') or, by (7)–(8'). When *all* the angles are required, the tangent formulas are better, since fewer logarithms are required than in (3), (4), (3'), (4'). It will be shown in Art. 69 that *r is the radius of the circle inscribed in the triangle.*

EXAMPLES.

1. In triangle ABC, $a = 25.17$, $b = 34.06$, $c = 22.17$. Find A, B, C.

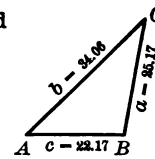

FIG. 60.

Formulas: $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

$$\tan \tfrac{1}{2}A = \frac{r}{s-a};\ \tan \tfrac{1}{2}B = \frac{r}{s-b};\ \tan \tfrac{1}{2}C = \frac{r}{s-c}.$$

$\therefore \log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) - \log s]$.

$\log \tan \tfrac{1}{2}A = \log r - \log(s-a)$; $\log \tan \tfrac{1}{2}B = \log r - \log(s-b)$;

$\log \tan \tfrac{1}{2}C = \log r - \log(s-c)$.

Check: $A + B + C = 180°$.

$a = 25.17$	$\log s = 1.60959$	$\log \tan \tfrac{1}{2}A = 9.64465 - 10$
$b = 34.06$	$\log(s-a) = 1.19117$	$\tfrac{1}{2}A = 23° 48' 28''$
$c = 22.17$	$\log(s-b) = 0.82217$	$\log \tan \tfrac{1}{2}B = 10.01365 - 10$
$2s = 81.40$	$\log(s-c) = 1.26788$	$\tfrac{1}{2}B = 45° 54'$
$s = 40.70$	$\therefore \log r^2 = 1.67163$	$\log \tan \tfrac{1}{2}C = 9.56794 - 10$
$s - a = 15.53$	$\therefore \log r = 0.83582$	$\tfrac{1}{2}C = 20° 17' 35''$
$s - b = 6.64$		
$s - c = 18.53$		

$\therefore A = 47° 36' 56''$, $B = 91° 48'$, $C = 40° 35' 10''$

Check: $A + B + C = 180° 0' 6''$.

2. Solve ABC, given $a = 260$, $b = 280$, $c = 300$.

3. Solve ABC when $a = 26.19$, $b = 28.31$, $c = 46.92$.

4. Solve PQR, given $p = 650$, $q = 736$, $r = 914$.

5. Solve RST, given $r = 1152$, $s = 2016$, $t = 2592$.

6. Solve Exs. 1, 4, Art. 58, using formulas (3)–(8′), without logarithms.

63. Problems in heights and distances. Some problems in heights and distances have been solved in Art. 29 by the aid of right-angled triangles. Additional problems of the same kind will now be given, in the solution of which oblique-angled triangles may be used. It is advisable to draw the figures neatly and accurately. The graphical method should also be employed.

EXAMPLES.

1. Another solution of Ex. 2, Art. 29.

In the triangle ABP (Fig. 23), $AB = 100$ ft., $BAP = 30°$, $PBA = 180° - 45° = 135°$. Hence the triangle can be solved, and BP can be found. When BP shall have been found, then in the triangle CBP, BP is known and $BP = 45°$; hence CP can be found. The computation is left to the student.

2. Another solution of Ex. 3, Art. 29. In the triangle CBP (Fig. 24), $BP = 30$ ft., $BCP = 40° 20' - 38° 20' = 2°$, $PBC = 90° + LCB = 128° 20'$. Hence CBP can be solved and the length of CB can be found. When CB shall have been found, then, in the triangle LCB, angle $C = 38° 20'$, CB is known, and hence LB can be found. The computation is left to the student.

3. Find the distance between two objects that are invisible from each other on account of a wood, their distances from a station at which they are visible being 441 and 504 yd., and the angle at the station subtended by the distance of the objects being 55° 40′.

4. The distance of a station from two objects situated at opposite sides of a hill are 1128 and 936 yd., and the angle subtended at the station by their distance, is 64° 28′. What is their distance?

5. Find the distance between a tree and a house on opposite sides of a river, a base of 330 yd. being measured from the tree to another station, and the angles at the tree and the station formed by the base line and lines in the direction of the house being 73° 15′ and 68° 2′, respectively. Also find the distance between the station and the house.

6. Find the height of a tower on the opposite side of a river, when a horizontal line in the same level with the base and in the same vertical plane with the top is measured and found to be 170 ft., and the angles of elevation of the top of the tower at the extremities of the line are 32° and 58°, the height of the observer's eye being 5 ft.

7. Find the height of a tower on top of a hill, when a horizontal base line on a level with the foot of the hill and in the same vertical plane with the top of the tower is measured and found to be 460 ft.; and at the end of the line nearer the hill the angles of elevation of the top and foot of the tower are 36° 24′, 24° 36′, and at the other end the angle of elevation of the top of the tower is 16° 40′.

8. A church is at the top of a straight street having an inclination of 14° 10′ to the horizon; a straight line 100 ft. in length is measured along the street in the direction of the church; at the extremities of this line the angles of elevation of the top of the steeple are 40° 30′, 58° 20′. Find the height of the steeple.

9. The distance between the houses C, D, on the right bank of a river and invisible from each other, is required. A straight line AB, 300 yd. long, is measured on the left bank of the river, and angular measurements are taken as follows: $ABC = 53° 30'$, $CBD = 45° 15'$, $CAD = 37°$, $DAB = 58° 20'$. What is the length CD?

10. A tower CD, C being the base, stands in a horizontal plane; a horizontal line AB on the same level with the base is measured and found to be 468 ft.; the horizontal angles BAC, ABC, are equal to 125° 40′, 12° 35′, respectively, and the vertical angles CAD, CBD, are equal to 38° 20′, 11° 50′, respectively. Find the height of the tower and its distances from A and B.

11. A base line AB 850 ft. long is measured along the straight bank of a river; C is an object on the opposite bank; the angles BAC, ABC, are observed to be 63° 40′, 37° 15′, respectively. Find the breadth of the river.

12. A tower subtends an angle α at a point on the same level as the foot of the tower and, at a second point, h feet above the first, the depression of the foot of the tower is β. Show that the height of the tower is $h \tan\alpha \cot\beta$.

13. The elevation of a steeple at a place due south of it is 45°, and at another place due west of it the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is $a(\sqrt{3}-1) \div 2\sqrt{2}$.

14. The elevation of the summit of a hill from a station A is α; after walking c feet toward the summit up a slope inclined at an angle β to the horizon the elevation is γ. Show that the height of the hill above A is $c \sin\alpha \sin(\gamma - \beta) \operatorname{cosec}(\gamma - \alpha)$ ft.

64. Summary. The preceding discussions on the solution of triangles have shown that a triangle may be solved in the following ways:

I. By the graphical method. [Arts. 10, 14, 21–24.]

II. If the triangle is right-angled, it can be solved, either with or without logarithms, by the methods shown in Arts. 25–27.

III. If the triangle is oblique, it can be divided into right-angled triangles, each of which can be solved by either of the methods II. [Art. 34.]

IV. The triangle, whether right-angled or oblique, can be solved without using logarithms, by means of formulas (1), (3), Art. 54; (1) or (2), Art. 61; (3)–(8), Art. 62.

V. The triangle, whether right-angled or oblique, can be solved with the use of logarithms, by means of formulas (1), Art. 54; (1) or (2), Art. 61; (3)–(8), Art. 62.

Checks: Any formula not employed in the computation can be employed as a check; that is, as a test for the correctness of the result.

Two things are necessary on the part of one who wishes to do well in the solution of triangles:

(1) The formulas referred to above should be clearly understood and readily derived.

(2) The arithmetical work required should be done accurately.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this chapter will be found at pages* 189–193.

CHAPTER VIII.

SIDE AND AREA OF A TRIANGLE. CIRCLES CONNECTED WITH A TRIANGLE.

65. Length of a side of a triangle in terms of the adjacent sides and the adjacent angles. In this proof, regard is paid to the conventions about signs, described in Arts. 36, 37. Let ABC be any triangle. From A draw AD perpendicular to BC, or BC produced. The positive direction of BC is in the direction of V. [At the first reading, only Fig. 61 a may be regarded.]

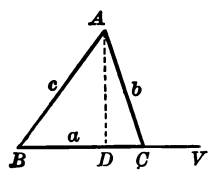

FIG. 61a.

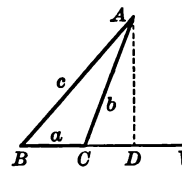

FIG. 61b.

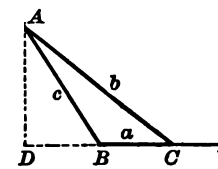

FIG. 61c.

$$BC = BD + DC,$$
$$= -CD + BD,$$
$$= -CA\cos VCA + BA\cos VBA. \quad \text{[Art. 40.]}$$

But $VCA = 180° - ACB.$

$\therefore \cos VCA = \cos(180° - ACB) = -\cos ACB.$ [Art. 45.]

$$\therefore BC = CA\cos ACB + BA\cos CBA;$$

i.e. $$a = b\cos C + c\cos B. \qquad (1)$$

Therefore, in any triangle each side is equal to the sum of the products of each of the other sides by the cosine of the angle which it makes with the first side.

When C is a right angle, (1) reduces to $a = c\cos B$.

EXAMPLE. Write the corresponding formulas for b and c. Derive these formulas.

66. Area of a triangle. Suppose that the area of a triangle ABC is required. Let the length of the perpendicular DC from

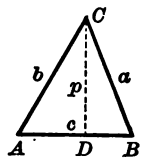

FIG. 62a.

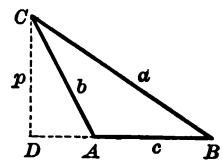

FIG. 62b.

C to AB, or AB produced, be denoted by p, and let the area be denoted by S. The following cases may occur:

I. *One side and the perpendicular on it from the opposite angle known*, say (c, p).

$$S = \tfrac{1}{2}cp. \quad \text{[By geometry.]} \tag{1}$$

II. *Two sides and their included angle known*, say, b, c, A. (See Figs. 62 *a*, 62 *b*.)

$$S = \tfrac{1}{2}cp = \tfrac{1}{2}c \cdot AC \sin BAC. \quad \text{[Art. 40.]}$$

$$\therefore S = \tfrac{1}{2}bc \sin A. \quad \text{[Compare Art. 31.]} \tag{2}$$

III. *Three sides known.*

$$S = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}bc \cdot 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A, \quad \text{[Art. 50.]}$$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}. \quad \text{[Art. 62.]}$$

$$\therefore S = \sqrt{s(s-a)(s-b)(s-c)}. \tag{3}$$

That is, *the area of a triangle is equal to the square root of the product of half the sum of the sides by the three factors formed by subtracting each side in turn from this half sum.* See Art. 34 *a* for another derivation of this formula.

IV. *One side and the angles known*, say, a, A, B, C.

$$S = \tfrac{1}{2}ab \sin C. \qquad \text{Now } b = \frac{a \sin B}{\sin A}.$$

$$\therefore S = \frac{1}{2}\frac{a^2 \sin C \sin B}{\sin A}. \tag{4}$$

EXAMPLE. Write and also derive the similar formulas in b and c.

EXAMPLES.

1. Find the areas of the triangles in Exs. 1–5, Art. 61.
2. Find the areas of the triangles in Exs. 1–5, Art. 62.
3. Find the areas of the triangles in Exs. 2, 3, Art. 60.

67. Area of a quadrilateral in terms of its diagonals and their angle of intersection.

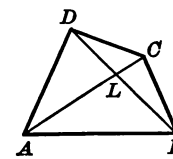

FIG. 63.

Area $ABCD$ = area ADC + area ABC.

Area ADC = area ALD + area CLD

$= \frac{1}{2} AL \cdot LD \sin ALD$

$+ \frac{1}{2} CL \cdot LD \sin CLD$ [Art. 66 (2).]

$= \frac{1}{2}(AL+LC) DL \sin ALD$, (since $\sin CLD = \sin ALD$)

$= \frac{1}{2} AC \cdot DL \sin ALD$.

Similarly,

area $ABC = \frac{1}{2} AC \cdot BL \sin BLC = \frac{1}{2} AC \cdot BL \sin ALD$.

$\therefore$ area $ABCD = \frac{1}{2} AC(DL+LB) \sin DLA = \frac{1}{2} AC \cdot BD \sin DLA$.

$\therefore$ area of a quadrilateral is equal to one-half the product of the two diagonals and the sine of their angle of intersection.

EXAMPLES.

1. Find the area of a quadrilateral whose diagonals are 108, 240 ft. long, and inclined to each other at an angle 67° 40′. Find the sides and angles of a parallelogram having these diagonals.

2. So also when the diagonals are 360, 570 ft. long, and their inclination is 39° 47′.

3. The diagonals of a parallelogram are 347 and 264 ft., and its area is 40,437 sq. ft. Find its sides and angles.

4. Solve an isosceles trapezoid, knowing the parallel sides $a = 682.7$ metres, $c = 1242.6$ metres, and the non-parallel equal sides $b = d = 986.4$ metres. Find the angles, the area, the lengths and angle of inclination of the diagonals.

68. The circumscribing circle. Let the radius of the circle described about a triangle ABC be denoted by R. It has been shown (see equation (2), Art. 54) that

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}. \qquad (1)$$

That is, *the radius of the circumscribing circle of any triangle is equal to half the quotient of any side by the sine of the opposite angle.*

From (2), Art. 66, $\sin A = \frac{2S}{bc}$. Substitution of this in the first of equations (1), gives

$$R = \frac{abc}{4S}.$$

69. The inscribed circle. Let the radius of the circle inscribed in a triangle ABC be denoted by r. Join the centre O and the points of contact L, M, N. By geometry, the angles at L, M, N are right angles. Draw OA, OB, OC.

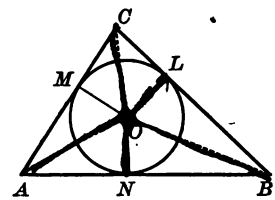

FIG. 64.

Area BOC + area COA + area AOB = area ABC.

$$\therefore \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \sqrt{s(s-a)(s-b)(s-c)}, \text{ or } S.$$

$$\therefore \tfrac{1}{2}(a+b+c)r = S,$$

i.e. $$sr = S.$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{S}{s}. \qquad (3)$$

That is, *the length of the radius of the inscribed circle of a triangle is equal to the number of units in its area divided by half the sum of the lengths of its sides.* See reference in Art. 62.

NOTE. Formula (8), Art. 62, can be readily derived from *Fig.* 64. By geometry, $AN = MA$, $BL = NB$, $CM = LC$.

Now $$\tan \tfrac{1}{2}A = \tan BAO = \frac{NO}{AN}.$$

But $NO = r$, and

$$AN = (AN + BL + CM) - (BL + LC) = s - a.$$

$$\therefore \tan \tfrac{1}{2}A = \frac{r}{s-a}.$$

70. The escribed circles. An *escribed circle* of a triangle is a circle that touches one of the sides of the triangle and the other two sides produced.

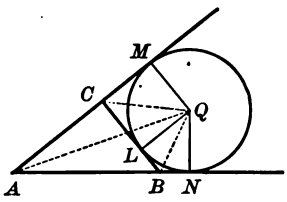

FIG. 65.

Let r_a denote the radius of the escribed circle touching the side BC opposite to the angle A. Join the centre Q and the points of contact L, M, N. By geometry, the angles at L, M, N are right angles. Draw QA, QB, QC.

Area ABQ + area CAQ − area BCQ = area ABC.

$$\therefore \tfrac{1}{2}r_a c + \tfrac{1}{2}r_a b - \tfrac{1}{2}r_a a = S,$$

$$\therefore \tfrac{1}{2}(c+b-a)r_a = S;$$

i.e.

$$(s-a)r_a = S.$$

$$\therefore r_a = \frac{S}{s-a}.$$

Similarly, $$r_b = \frac{S}{s-b}; \quad r_c = \frac{S}{s-c}.$$

Other interesting relations between the sides, angles, and related circles, of a triangle, are indicated in the exercises in the latter part of the book.

EXAMPLES.

1. Find the radii of the circumscribed, inscribed, and escribed circles of some of the triangles in Arts. 55–58.

2. Find the radii of these related circles of some of the triangles in Exs. 1–3, Art. 66.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this Chapter will be found at pages* 193, 194.

CHAPTER IX.

RADIAN MEASURE.

71. The radian defined. The system of measuring angles with a degree as the unit angle, was described in Art. 11. Since the time of the Babylonians this system has been the common practical method employed. Another method of measuring angles was introduced early in the last century. This method is used to some extent in practical work, and is universally used in the higher branches of mathematics. It is employed, on account of

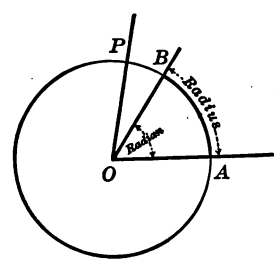

FIG. 66.

its great convenience, in the larger and more important part of what is now called trigonometry, namely, the part which is not concerned with the measurement of lines and angles, but which pursues investigation of the properties of the quantities that, so far in this book, have been called the trigonometric ratios. A very little knowledge of the trigonometric ratios is sufficient for the solution of triangles. The more detailed and extended study of angles and their six related numbers, constitutes part of what is sometimes called *Higher Trigonometry*, but, more generally, *Analytical Trigonometry*. This subject is a large one, and has close connections with many other branches of modern mathematics.

The system of angular measurement now to be described, is sometimes referred to as the *theoretical system* of measurement. **In this system the unit angle is the angle which at the centre of a circle subtends an arc equal in length to the radius.** This unit angle is called **a radian.** Thus, if a circle with any radius be described about O as a centre, and an arc AB be taken equal in length to the radius, then the angle AOB is a radian.

72. The value of a radian. In order that a quantity may be used as a unit of measurement, it must have a fixed value; that is, using the customary mathematical phrase, it must be a *constant* quantity. The proof that a radian has a fixed value, or is a constant quantity, depends upon two geometrical facts, viz.:

(a) In the same circle two angles at the centre are in the same ratio as their intercepted arcs.

(b) The ratio of a circumference of a circle to its diameter is the same for all circles. [See Art. 9 (b).]

For the proof of (a), reference may be made to any plane geometry; for instance, to Euclid VI., 33.* The proof of (b) is not contained in all geometries; for instance, Euclid does not give it.† Accordingly, an outline of such a proof and the calculation of π are given in Note C of the Appendix. This note should now be studied by those whose course in plane geometry has not included

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* The truth of theorem (a) can easily, by an inductive method, be made evident to students who have not proved the theorem in plane geometry. Thus, on taking angles which are twice, three times, four times, one-half, one-third, etc., of a given angle, it can be seen that their respective arcs bear the same relations to one another.

† Euclid lived about 323–283 B.C. Archimedes (287 ?–212 B.C.), the greatest mathematician of antiquity, measured the length of the circle and the area contained by it, and also measured the surface of the sphere. He showed that the ratio of the circle to its diameter lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. In 1794 a French mathematician, Adrien Marie Legendre (1752–1833), published his *Elements of Geometry*, in which the works of Euclid and Archimedes on elementary geometry are blended together. The elementary text-books now in use on the continent of Europe and in the United States, are written mainly on Legendrean lines; the geometrical text-books generally studied throughout the British Empire, are editions of **Euclid's *Elements*.**

the measurement of the circle. Theorems (*a*) and (*b*) are assumed in what follows.

In Fig. 66, $\dfrac{\textit{the radian } AOB}{4 \textit{ right angles}} = \dfrac{\textit{arc } AB}{\textit{circumference of circle}}$ [By (*a*).]

$$= \frac{r}{2\pi r} = \frac{1}{2\pi} \qquad \text{[By (b).]}$$

$$\therefore \text{ the radian} = \frac{1}{2\pi} \times 4 \text{ right angles} = \frac{2}{\pi} \times \text{right angle.} \qquad (1)$$

Since all right angles are equal, and since each radian is a fixed fraction, namely, $\frac{2}{\pi}$, of a right angle, it follows that *all radians are equal.* It will be remembered that the unit in the common practical system is one-ninetieth of a right angle.

From (1),

$$\text{A radian} = \frac{180^\circ}{\pi} \qquad (2)$$

$$= \frac{180^\circ}{3.14159\ldots} = 57^\circ\ 17'\ 44''.81 \text{ approximately} *$$

$$= 206265'' \text{ approximately.}$$

Ex. With a protractor lay off an angle approximately equal to a radian. Compare it with angle 60°. An angle 60°, at centre of a circle, is subtended by *a chord* equal in length to the radius; a radian is subtended by *an arc* equal in length to the radius.

73. The radian measure of an angle. Measure of a circular arc. *The radian measure of an angle is the ratio of the angle to a radian.* [See Art. 8.] For instance, if an angle A is twice a radian, then its radian measure is 2; if an angle B is two-thirds of a radian, then its radian measure is $\frac{2}{3}$. This is expressed thus:

$$A = 2 \text{ radians} = 2^r; \quad B = \tfrac{2}{3} \text{ radians} = \tfrac{2}{3}^r. \qquad (3)$$

Here, r is used as the symbol for radians just as ° is used as the symbol for degrees in 23°. In general discussions the radian

* The value of the radian has been calculated by J. W. L. Glaisher to 41 places of decimals of a second. [*Proc. Lond. Math. Soc.*, Vol. IV. (1871–73), pp. 308–312.]

measure of an angle is often expressed by Greek letters; thus, the angles α, β, θ, ϕ, etc., contain α, β, θ, ϕ, etc., radians. In these cases the symbol r is usually omitted, but it is always *understood* that the radian is the unit of measurement.

If the circular arc subtended by an angle is equal in length to *twice* the radius, then the radian measure of the angle is obviously *two;* if the arc is *one-half* the length of the radius, then the angle contains *half* a radian. *The radian measure of an angle may be given* **a second definition,** which depends on Theorem (a), Art. 72. Let AOP, Fig. 66, be any angle, and AOB be a radian. Describe a circle with any radius OA, equal to r, about the vertex O as a centre. Let arc AB be equal to the radius, and draw OB. Then angle AOB is a radian, by the definition in Art. 71.

Now, $$\frac{\text{angle } AOP}{\text{radian } AOB} = \frac{\text{arc } AP}{\text{arc } AB} \quad \text{[Th. } (a)\text{, Art. 72.]} \qquad (4)$$

i.e. $$\frac{\text{angle } AOP}{\text{the radian}} = \frac{\text{arc } AP}{\text{the radius}}. \qquad (5)$$

That is, the number of radians in an angle, or **the radian measure of an angle,** is the answer to the question: **how many times does any circular arc subtended by it, contain the radius?** Thus, for example, the radian measures of the angles which subtend circular arcs equal in length to 2, 3, 1.5, .825 radii are 2, 3, 1.5, .825, respectively.

The circular arc subtended by $360° = 2\pi r$;

hence, $$\text{radian measure of } 360° = \frac{2\pi r}{r} = 2\pi,$$

and $$\text{radian measure of } 180° = \pi. \qquad (6)$$

This shows that an angle 2π radians is described each time that the revolving line makes a complete revolution. Relation (6), namely,

$$180° = \pi \text{ radians}, \qquad (7)$$

connects the two systems of angular measurement. By means of (7), an angle expressed in the one system can be expressed in the other. The word *radians* is usually omitted from (7), but is always understood. Relation (7) may also be deduced directly

from (1). Just as angles are considered as unlimited in magnitude, so arcs are considered as unlimited in length.

NOTE 1. The term *circular measure* is often used for *radian measure*, and c is used as the symbol for radians. Thus (3) is written $A = 2^c$, $B = \frac{1}{2}^c$.

Ex. 1. Express 30° in radian measure.

Since $$180° = \pi, \quad 1° = \frac{\pi}{180};$$

$$30° = \tfrac{30}{180}\pi = \frac{\pi}{6}. \quad \text{Also, } 30° = \frac{\pi}{6} = \frac{3.14159^r \cdots}{6} = .52359^r \cdots.$$

The term *radians* and the symbol for radian is usually omitted from the second members of these equations, but is always *understood* to be there.

Ex. 2. Express 45°, 60°, 135°, 210°, 300°, 330°, 270°, 225°, − 75°, 63°, 27°, − 33°, − 150°, in radian measure, (*a*) as fractions of π, (*b*) numerically, on putting $\pi = \frac{22}{7}$.

Ex. 3. Express the angle $\frac{9}{10}\pi$ in degrees.

Here, "$\frac{9}{10}\pi$" means "$\frac{9}{10} \times \frac{22}{7}$ *radians*."

Since $$\pi = 180°, \quad \tfrac{9}{10}\pi = \tfrac{9}{10} \times 180° = 162°.$$

Ex. 4. Express the angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$, $\frac{\pi}{5}$, $\frac{2}{3}\pi$, $\frac{7}{8}\pi$, 10π, 4π, 3π, 6π, $\frac{5}{2}\pi$, $\frac{5}{3}\pi$, in degrees, and their complements and supplements, in radians.

Ex. 5. Express the angles $-\frac{1}{4}\pi$, -5π, $-\frac{4}{3}\pi$, $-\frac{11}{12}\pi$, -25π, in degrees.

Ex. 6. Express 2^r (2 radians) in degrees.

Since $$\pi = 180°,$$

$$\therefore 1^r = \frac{180°}{\pi};$$

$$\therefore 2^r = 2 \times \frac{180°}{\pi} = 114° 35' 29.6'' \text{ approximately.}$$

Ex. 7. Express $\frac{1^r}{2}$, 4^r, 3^r, $\frac{1^r}{3}$, $\frac{1^r}{5}$, 5^r, 10^r, $\frac{1^r}{10}$, in degrees.

Measure of an arc. Since, by (5),

$$\frac{\text{subtended circular arc}}{\text{radius}} = \text{number of radians in the angle,}$$

then **length of arc = radius × number of radians in the angle.**

If a denote the length of any arc AP, r the radius, θ the radian measure of angle AOP, then

$$a = r\theta. \tag{8}$$

In words: **The length of any circular arc is equal to the product of the radius and the radian measure of its subtended central angle.** For example, the arc of $360° = 2\pi$ radii, arc of $180° = \pi$ radii, etc. These arcs are usually referred to as the arcs 2π, π, etc.; but it is always *understood* that *the radius is the unit of measurement.* The symbol π, which always denotes the incommensurable number 3.14159 ···, can thus be used in *three* connections in trigonometry:

(1) With other numbers, as a number simply.

(2) With reference to *angles;* in which case it denotes an angle containing π radians, *i.e.* 3.14159 ··· radians.

(3) With reference to *arcs;* in which case it denotes an arc containing 3.14159 radii. This is an arc subtended by a central angle of π radians.

The expression $180° = \pi$ does not mean $180° = 3.14159 \cdots$; it means $180° = 3.14159$ *radians.*

The expression "arc π" does not mean arc 3.1416; it means "arc of 3.1416 *radii.*" In any particular instance, the context will show to what π refers, whether to *angle* or *arc.*

It is evident from the second definition of radian measure that, like the trigonometric ratios, **the radian measure of an angle is also a ratio of one line to another,** namely, the ratio of the subtended circular *arc* to its *radius.*

NOTE 2. *If the radius be taken as unit length, then, by* (8) *or* (5), *the number of units of length in the arc is the same as the number of radians in the angle.*

EXAMPLES.

8. What is the radian measure of the angle which at the centre of a circle of radius $1\frac{1}{2}$ yd. subtends an arc of 8 in.? Also express the angle in degrees.

Let θ denote the radian measure of the angle. Then

$$\theta = \frac{\text{arc}}{\text{rad}} = \frac{8 \text{ in.}}{1.5 \text{ yd.}} = \frac{8}{54} = \frac{4}{27}.$$

Since $\pi = 180°$,

$$\therefore 1^r = \frac{180}{\pi};$$

$$\therefore \left(\tfrac{4}{27}\right)^r = \tfrac{4}{27} \times \tfrac{7}{22} \times 180° = 8° 29' \text{ approximately.}$$

9. Give the trigonometric ratios of

$$\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \tfrac{5}{4}\pi, -\tfrac{5}{3}\pi, -\tfrac{7}{4}\pi, -\tfrac{5}{6}\pi.$$

10. Find the numerical values of (*a*) $\sin^2\frac{\pi}{6} + \cos^2\frac{7}{4}\pi + \tan^2\frac{2\pi}{3}$,

(*b*) $3\sin\frac{5}{6}\pi\cos\frac{17}{3}\pi\tan\frac{23}{4}\pi$, (*c*) $2\sin\frac{23}{3}\pi\cos\frac{23}{6}\pi\tan\frac{23}{3}\pi$.

11. Find the number of radians (*a*) as fractions of π, (*b*) numerically (on putting $\pi = \frac{22}{7}$), in each interior and exterior angle of the following regular polygons: pentagon, hexagon, heptagon, octagon, decagon, dodecagon, quindecagon.

12. Find the number of radians and the number of degrees in the following angles subtended at the centres of circles: (1) arc 10 in., radius 3.5 in.; (2) arc $\frac{1}{7}$ ft., radius 2 ft.; (3) arc 1 mi., radius 7920 mi.; (4) arc 250 mi., radius 8000 mi.; (5) arc 10 yd., radius 10 mi.; (6) arc $\frac{1}{10}$ mi., radius 10 ft.

13. What are the radii when an arc 10 in. in length subtends central angles containing 1, 2, 4, 6, 8, 12, 15, 20, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{3}$, radians respectively?

14. What are the radii when an arc 10 in. in length subtends central angles containing 1°, 2°, 3°, 16°, 28°, 120°, 30′, 20′, 10′, 10″, 20″, 45″, respectively?

15. In a circle whose radius is 10 in., what are the lengths of the arcs subtended by central angles containing 1, 4, 7, 8, 12, .5, .375, .125, radians respectively?

16. In the circle in Ex. 15, what are the lengths of the arcs subtended by central angles containing 2°, 25°, 48°, 135°, 250°, 30′, 45′, 30″, 50″, respectively?

17. What are the areas of the circular sectors in Exs. 13, 15? [See Note C, 5.]

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this Chapter will be found at pages* 194, 195.

CHAPTER X.

ANGLES AND TRIGONOMETRIC FUNCTIONS.

74. Chapters II., V., contain little more about the trigonometric ratios than is needed in the solution of triangles. In this and the following chapters a further study of these ratios is made. Although the results of this study are not applicable to such ordinary practical uses as the measurement of triangles, heights, and distances, yet they are very interesting in themselves, and help to give a better and fuller understanding of the connection between angles and trigonometric ratios. These results are also useful in further mathematical work, and in the study of various branches of mechanical and physical science. In reading Chapters X., XI., acquaintance will be made, or renewed, with some important general ideas of mathematics.

75. Function. Trigonometric functions. If a number is so related to one or more other numbers, that its values depend upon their values, then it is *a function* of these other numbers. Thus the circumference of a circle is a function of its radius; the area of a rectangle is a function of its base and height; the area of a triangle is a function of its three sides.

NOTE. The values of such expressions as $2x - 5$, $x^2 - 4x + 7$, $\log_{10} x$, 2^x, depend upon the values given to x. These expressions are, accordingly, functions of x. A function of x is usually denoted by one of the symbols $f(x)$, $F(x)$, $\phi(x)$, etc., which are read "the f-function of x," "the F-function of x," "the *Phi*-function of x," etc.

The trigonometric ratios of an angle depend upon the value (*i.e.* magnitude) of the angle. On this account the trigonometric ratios are very often called the *trigonometric functions.* They are also frequently called the *circular functions.*

The trigonometric (or circular) functions include not only the

six functions previously discussed, namely, *sine, cosine, tangent, cotangent, secant, cosecant,* but also three others, viz.:

versed sine of $A = 1 - \cos A$, written **vers** A,

coversed sine of $A = 1 - \sin A$, written **covers** A,

suversed sine of $A = 1 + \cos A$, written **suvers** A.

The versed sine is used not unfrequently; the latter two are rarely used.

EXAMPLES.

Find the remaining eight trigonometric functions when:

1. $\sin A = .3$. 2. $\cos A = .4$. 3. $\tan A = -3$. 4. $\cot A = .7$.
5. $\sec A = -3$. 6. $\operatorname{cosec} A = .8$. 7. $\operatorname{vers} A = 1.5$. 8. $\operatorname{vers} A = .5$.

9. Show that $\dfrac{\sqrt{2 \operatorname{vers} A - \operatorname{vers}^2 A}}{1 - \operatorname{vers} A} = \tan A$.

10. Show that $\cos\theta \operatorname{vers}\theta (1 + \sec\theta) = \sin^2\theta$.

76. Algebraical note.

It will be useful to have an idea of the meaning of the word *limit* as used in mathematics. In the geometrical series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots,$$

the sum of 2 terms is $1\frac{1}{2}$, of 3 terms is $1\frac{3}{4}$, of 4 terms is $1\frac{7}{8}$, of 5 terms is $1\frac{15}{16}$. The sum of the series varies with the number of terms taken; and the greater the number of terms taken, the more nearly does their sum approach 2. It is stated in arithmetic and algebra that the sum of an infinitely great number of terms of this series is $1 \div (1 - \frac{1}{2})$, *i.e.* 2. This simply means that, by making the number of terms as great as one please, the sum can be made to approach as nearly as one please to 2; or, in other words, the greater the number of terms taken, the more nearly does their sum approach the value 2. This idea is expressed in mathematics in slightly different language: "The *limit* of the sum of this series is 2." In geometry (see Note C) it is shown that if a regular polygon be inscribed in a circle, the length of the perimeter of the polygon approaches nearer and nearer to the length of the circle as the number of the sides of the polygon is increased; also the area of the polygon approaches nearer and nearer to the area of the circle. The length of the circle is said to be the *limit* of the length of the perimeter of the inscribed polygon, and the area of the circle is said to be the *limit* of the area of the polygon, as the number of its sides is indefinitely increased.

Definition. If a varying quantity approaches nearer and nearer to a fixed quantity (or given *constant*), so that the difference between the two quanti-

ties may become, and remain, as small as one please, then the fixed quantity is called *the limit* of the varying quantity.

The following algebraic principles are required in some of the articles that follow:

(*a*) If the numerator of a fraction is finite, and its absolute value either remains constant or increases while the denominator decreases, then the absolute value of the fraction increases. Thus, if in $\frac{a}{x}$, a either remains constant or increases while x decreases, then $\frac{a}{x}$ increases; *e.g.*

$$\frac{a}{1}, \frac{a}{\frac{1}{10}}, \frac{a}{\frac{1}{100}}, \frac{a}{\frac{1}{1000}}, \cdots, \text{ have values } 10\,a,\ 100\,a,\ 1000\,a, \ldots.$$

It is also evident that the smaller x becomes, the larger does $\frac{a}{x}$ become; and that $\frac{a}{x}$ approaches an exceedingly great value when x approaches zero. In other words, when the number x approaches the value zero as its limit, then the number $\frac{a}{x}$ approaches an inconceivably great value as *its* limit. In mathematics numbers of the latter kind are each denoted by the word *infinity* and by the symbol ∞; that is, the symbol ∞ denotes any number which is greater than any number that can be assigned. The principle just stated, may be briefly expressed:

$$\text{If } x \doteq 0, \text{ then } \frac{a}{x} \doteq \infty,$$

in which the symbol $\doteq$ indicates the following reading: If x approaches zero as a limit, then $\frac{a}{x}$ approaches infinity as a limit. The same idea is also expressed thus:

$$\operatorname*{Limit}_{x=0} \frac{a}{x} = \infty;$$

i.e. the limit of $\frac{a}{x}$, when x is zero, is infinity.

(*b*) It is also evident that as x increases, $\frac{a}{x}$ decreases (a remaining finite); and that when x approaches an infinitely great value, $\frac{a}{x}$ approaches zero. That is, if

$$x \doteq \infty, \text{ then } \frac{a}{x} \doteq 0; \text{ or, } \operatorname*{Limit}_{x=\infty} \frac{a}{x} = 0.$$

(*c*) If x approaches zero, then $\frac{x}{a}$ approaches zero, so long as a does not approach zero; that is,

$$\operatorname*{Limit}_{x=0} \frac{x}{a} = 0.$$

77. Changes in the trigonometric functions as the angle increases from 0° to 360°. For convenience the revolving line will be kept constant in length in the following explanations. The student should try to deduce the changes in the functions for himself, especially after reading about the changes in the sine.

Change in sin A as A increases from 0° to 360°.

If OP be any position of the revolving line, then

$$\sin XOP = \frac{MP}{OP}.$$

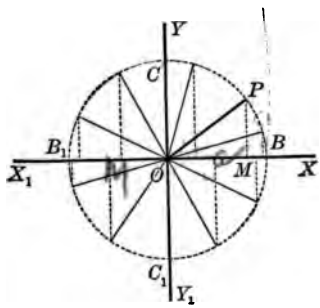

FIG. 67.

Now OP is kept the same in length, say length a, as XOP increases from 0° to 360°. Hence, in order to trace changes in the sine as the angle changes, it is necessary to consider only the changes in MP. Let the angle be denoted by A.

When $A = 0$, OP coincides with OB, and $MP = 0$. $\therefore \sin 0° = \frac{0}{a} = 0.$

As OP revolves from OX to OY, MP increases in length and is positive.

When $A = 90°$, OP coincides with OC, and $MP = a$. $\therefore \sin 90° = \frac{a}{a} = 1.$

Hence, as the angle A increases from 0° to 90°, its sine increases from 0 to 1.

As OP revolves from OY to OX_1, MP decreases in length and is positive.

When $A = 180°$, OP coincides with OB_1, and $MP = 0$. $\therefore \sin 180° = \frac{0}{a} = 0.$

Hence, as the angle A increases from 90° to 180°, its sine decreases from 1 to 0.

As OP revolves from OX_1 to OY_1, MP increases in length and is negative; *i.e.* MP really decreases.

When $A = 270°$, OP coincides with OC_1, $MP = a$ and is negative.

$$\therefore \sin 270° = \frac{-a}{a} = -1.$$

Hence, as the angle A increases from 180° to 270°, its sine decreases from 0 to -1.

As OP revolves from OY_1 to OX, MP decreases in length and is negative; *i.e.* MP really increases.

When $A = 360°$, OP coincides with OB, and $MP = 0$. $\therefore \sin 360° = \frac{0}{a} = 0$.

Hence, as the angle A increases from 270° to 360°, its sine increases from -1 to 0.

If OP continues to revolve, then the sine again undergoes the same changes in the same order, and does so during each successive revolution.

Change in cos A as A increases from 0° to 360°.

In Fig. 67, $\cos XOP = \frac{OM}{OP}$. Hence, in order to trace the changes in the cosine as the angle changes, it is necessary to consider only the changes in OM, since OP is kept at a constant length a.

When $A = 0°$, OP coincides with OB, and $OM = a$. $\therefore \cos 0° = \frac{a}{a} = 1$.

As OP revolves from OX to OY, OM decreases in length and is positive.

When $A = 90°$, OP coincides with OC, and $OM = 0$. $\therefore \cos 90° = \frac{0}{a} = 0$.

Hence, as the angle A increases from 0° to 90°, its cosine decreases from 1 to 0.

As OP revolves from OY to OX_1, OM increases in length and is negative, *i.e.* OM really decreases.

When $A = 180°$, OP coincides with OB_1, $OM = a$, and is negative.

$$\therefore \cos 180° = \frac{-a}{a} = -1.$$

Hence, as the angle increases from 90° to 180°, its cosine decreases from 0 to -1.

On proceeding in the same manner, the student will discover that:

As A increases from 180° to 270°, $\cos A$ increases from -1 to 0;

As A increases from 270° to 360°, $\cos A$ increases from 0 to 1.

If OP continues to revolve, then the cosine again undergoes the same changes in the same order, and does so during each successive revolution

Change in tan A as A increases from 0° to 360°.

In Fig. 67, $\tan XOP = \frac{MP}{OM}$. Hence, in order to trace the changes in the tangent as the angle changes, it is necessary to consider the changes in MP and OM.

When $A = 0°$, OP coincides with OB, $MP=0$, $OM=a$. $\therefore \tan A = \frac{0}{a} = 0$.

As OP revolves from OX to OY, MP increases and OM decreases, and both are positive; hence, $\tan A$ increases.

When $A = 90°$, OP coincides with OC, $MP = a$, $OM = 0$.

$$\therefore \tan 90° = \frac{a}{0} = \infty.$$

As OP revolves from OY to OX_1, MP decreases and is positive, OM increases in length and is negative; hence, $\tan A$ decreases in magnitude and is negative; *i.e.* $\tan A$ really increases. [When OP passes at OY from the first quadrant into the second, the value of the tangent changes from $+\infty$ to $-\infty$, for OM changes its sign from $+$ to $-$.]

When $A = 180°$, OP coincides with OB_1, $MP = 0$, $OM = -a$.

$$\therefore \tan 180° = \frac{0}{-a} = 0.$$

Hence, as A increases from 90° to 180° $\tan A$ increases from $-\infty$ to 0.

On proceeding in the same manner the student will discover that:

As A increases from 180° to 270°, $\tan A$ increases from 0 to $+\infty$;

As A increases from 270° to 360°, $\tan A$ increases from $-\infty$ to 0.

If OP continues to revolve, then the tangent again undergoes the same changes in the same order, and does so during each successive revolution.

In the same way as above, the student can trace the changes in *sec A*, *cosec A*, *cot A*, as A increases from 0° to 360°. The changes in these functions can also be deduced from the results obtained for $\sin A$, $\cos A$, $\tan A$, and the relations

$$\sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}, \quad \cot A = \frac{1}{\tan A}.$$

The results are collected in the following table: *

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* This method of indicating the changes in the trigonometric functions is that given in Loney's *Plane Trigonometry*, p. 57.

Y

In the second quadrant the					In the first quadrant the				
sine	decreases from	1	to	0	sine	increases from	0	to	1
cosine	decreases from	0	to	-1	cosine	decreases from	1	to	0
tangent	increases from	$-\infty$	to	0	tangent	increases from	0	to	∞
cotangent	decreases from	0	to	$-\infty$	cotangent	decreases from	∞	to	0
secant	increases from	$-\infty$	to	-1	secant	increases from	1	to	∞
cosecant	increases from	1	to	∞	cosecant	decreases from	∞	to	1

X_1 —— O —— X

In the third quadrant the					In the fourth quadrant the				
sine	decreases from	0	to	-1	sine	increases from	-1	to	0
cosine	increases from	-1	to	0	cosine	increases from	0	to	1
tangent	increases from	0	to	∞	tangent	increases from	$-\infty$	to	0
cotangent	decreases from	∞	to	0	cotangent	decreases from	0	to	$-\infty$
secant	decreases from	-1	to	$-\infty$	secant	decreases from	∞	to	1
cosecant	increases from	$-\infty$	to	-1	cosecant	decreases from	-1	to	$-\infty$

Y_1

NOTE. It should be observed that *the algebraic sign of each function changes when the function passes through either of the values zero and infinity.*

EX. Trace the changes in the versed sine as the angle changes from 0° to 360°.

78. Periodicity of the trigonometric functions. It has been seen in Arts. 40–44 that all angles coterminal with XOP have the same ratios. That is, the same ratios as XOP has, are obtained each time that the revolving line returns to the position OP, no matter how many complete revolutions in the positive or negative direction it may make in the meantime. In the last article it was pointed out that *the sine*, for instance, *always goes through all its changes* (the *cycle* of changes, namely, 0 to 1, 1 to 0, 0 to -1, -1 to 0) *in the same order when the turning line revolves from the position* OX *through the angle* 360° *or* 2π. According to the opening remarks of this article, *all angles which differ by any integral multiple* (positive or negative) *of* 2π *radians have the same sine.* These facts are expressed mathematically by saying:

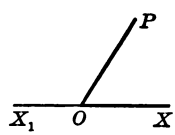


FIG. 68.

The sine is a periodic function, and **the period of the sine is 2π.**

Similar considerations show that *the cosine, secant, cosecant, are periodic functions, and that each of them has a period* 2π.

The *tangent* and *cotangent*, however (Art. 77), go through all their changes, while the angle increases by 180° or π radians. Hence, *the period of the tangent and cotangent is* π.

NOTE 1. These properties may be expressed as follows, m denoting any positive or negative whole number, and x being any angle:

$$\sin x = \sin(2m\pi + x), \quad \cos x = \cos(2m\pi + x),$$

and similar for sec x, cosec x;

$$\tan x = \tan(m\pi + x), \quad \cot x = \cot(m\pi + x).$$

NOTE 2. (*Algebraic.*) When a function $f(x)$ has the property that $f(x) = f(x + k)$, in which x can have any value and k is a constant, the function $f(x)$ is said to be a *periodic function*. If k is the *least quantity* for which this equation is true, then k is called *the period of the function*.

If $f(x) = f(x + k)$, then $f(x) = f(x + nk)$, n being any positive or negative whole number. For $f(x + k) = f(x + k + k) = f(x + 2k)$, and so on. Also, since $f(x) = f(x + k)$ for all values of x, this equation holds when $x - k$ is put for x; that is, $f(x - k) = f(x)$. Similarly, $f(x - 2k) = f(x - k) = f(x)$, and so on.

NOTE 3. It has been shown above that each of the trigonometric functions has but *one* period, namely, π, in the case of the tangent and cotangent, and 2π in the case of each of the other functions. Hence, the trigonometric functions are *singly periodic functions*. Functions which have more than one period appear in some branches of higher mathematics. For instance, certain functions called *elliptic functions* have two periods, and, accordingly, are said to be *doubly periodic*. *See Questions on Chap. X., Ex. 16.*

79. The old or line definitions of the trigonometric functions. The trigonometric functions were formerly considered as belonging to arcs rather than to angles, and were certain *lines* related to these arcs. Let APB be a circle described with any radius R about O as a centre. Let OA, OB, be at right angles to each other, and let AP be any arc having A for its initial point. Draw OP; from P draw PM at right angles to OA; through A draw a tangent AT to meet OP produced in T; through B draw a tangent BT_1 to meet OP produced in T_1; from P draw PM_1 at right angles

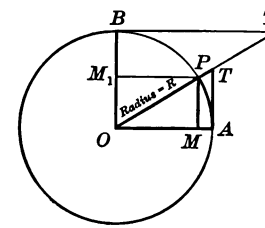


FIG. 69.

to *OB*. The lines *MP, AT, OT, AM,* were called respectively, the *sine, tangent, secant, coversed sine, of the arc AP;* and M_1P, BT_1, OT_1 (the sine, tangent, secant, of the complementary arc *PB*) were called respectively, the *cosine, cotangent, cosecant, of the arc AP*. These definitions are expressed in words as follows:

The *sine of an arc* is a straight line drawn from one extremity of the arc perpendicular to the radius passing through the other extremity. The *tangent of an arc* is a straight line touching the arc at one extremity, and limited by the radius produced through the other extremity. The *secant of an arc* is the straight line joining the centre of the circle, and the further extremity of the tangent drawn at the origin of the arc.

The *sine, tangent, and secant of the complement of an arc* are called the *cosine, cotangent,* and *cosecant* of that arc.

Since the arc measures the angle at the centre (the number of degrees in this arc is the same as the number of degrees in the subtended angle), these lines were also called the *sine, cosine,* ... of the central angle *AOP* measured by the arc *AP*. These lines were known as *the trigonometric lines.*

NOTE. By "the length of a line" is meant the *number* of units of length which it contains. The lengths of these lines *depend on the length of the radius of the circle*, as well as on the magnitude of the central angle subtended by the arc. Hence it was *necessary to specify the radius* when the functions were discussed. This inconvenience has led to the adoption of the ratio definitions.

In Fig. 69 let R denote the length of the radius. Then, on using the *ratio* definitions,

$$\sin AOP = \frac{MP}{R}, \tan AOP = \frac{AT}{R}, \sec AOP = \frac{OT}{R}.$$

Hence, the *ratio* definitions of the trigonometric functions can be derived from the *line* definitions by dividing the lengths of the lines by the length of the radius. *If the length of the* **radius is unity,** *then the lengths of the* **lines** *used in the line definitions are* **equal to the ratios** *in the ratio definitions.* This suggests a geometrical or graphical method of representing the trigonometric functions, which is shown in the next article.

80. Geometrical representation of the trigonometric functions. Let a circle of radius equal to unity be drawn. This circle is called a **unit-circle.** Let the construction described in Art. 79 be made for each of the angles AOP_1, AOP_2, AOP_3, ···. *In any circle* the lines M_1P_1, M_2P_2, M_3P_3, ···, are proportional to, and hence *represent* the sines of these angles,

the lines AT_1, AT_2, AT_3, ···,
represent the tangents of these angles,

the lines OT_1, OT_2, OT_3, ···,
represent the secants of these angles,

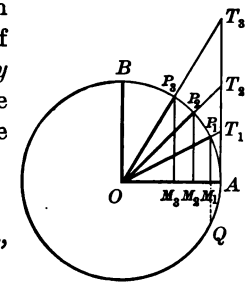


FIG. 70.

and so on for the other ratios. *In the unit-circle,* however, *the measures of these lines,* the radius being the unit of length, *are the very same numbers as the respective ratios* mentioned. In the unit-circle also, *the linear measure of the arc is the same as the radian measure of the angle which it subtends.* [See Art. 73, Note 2.]

SUGGESTED EXERCISES. (1) By means of *the lines* on the unit-circle, trace the changes in the trigonometric functions as the angle changes from 0° to 90°. Compare the results with those of Art. 77.

(2) For particular values of the angle AOP_1, measure the lengths of the related lines on the unit-circle, and compare the results with the values given in the tables of natural sines and tangents.

NOTE 1. The origin of the term *circular* functions, *tangent*, *secant*, is apparent from Art. 79.

NOTE 2. The name *sine* comes from the Latin word *sinus*, which was the translation of the Arabic word for this trigonometric function. The Arabic word for the sine resembled a word meaning an indentation or gulf.

NOTE 3. In trigonometry the Greeks used the *whole* chord P_1Q instead of the *half*-chord or sine. For example, Ptolemy, the celebrated astronomer who flourished about 125–151 A.D., gave a table of chords in Book I. of the *Almagest*, his work on astronomy. The Hindoos, on the other hand, always used the half-chord or sine. The Arabian astronomer, Al Battani (or Albatagnius) (877–929), in his work *The Science of the Stars*, like the Hindoos determined angles "by the semi-chord of twice the angle," *i.e.* by the *sine* of the angle, taking the radius as unity. The translation of this work into Latin in the twelfth century introduced the word *sine* into trigonometry. The *Hindoo sine* was finally adopted in Europe in preference to the *Greek chord* in the fifteenth century. [See Art. 12, foot-note.]

81. Graphical representation of functions.

Graphical representation. The different values which a varying quantity takes, are often represented by means of a curve. Many illustrations can be given of the graphical representation of various things whose values can be denoted by means of *numbers*. For example, the curve in Fig. 71 shows the record of the barometer at Ithaca from May 22 to May 29, 1899.

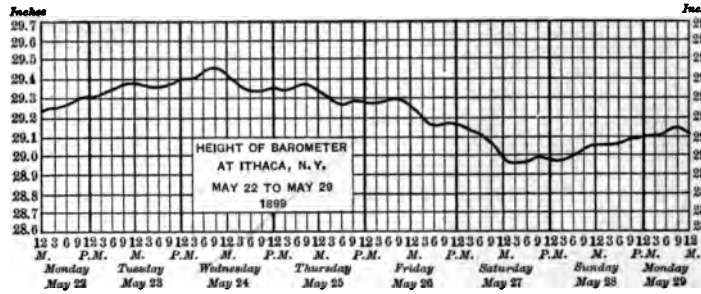
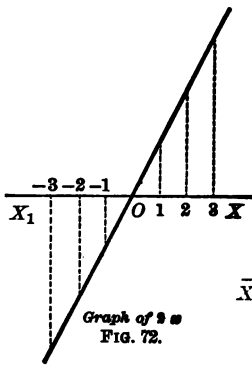

Inches
29.7
29.6
29.5
29.4
29.3
29.2
29.1
29.0
28.9
28.8
28.7
28.6
HEIGHT OF BAROMETER
AT ITHACA, N.Y.
MAY 22 TO MAY 29
1899
12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12 3 6 9 12
M. P.M. M. P.M. M. P.M. M. P.M. M. P.M. M. P.M. M. P.M. M.
Monday May 22
Tuesday May 23
Wednesday May 24
Thursday May 25
Friday May 26
Saturday May 27
Sunday May 28
Monday May 29

FIG. 71.

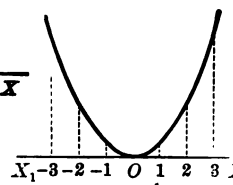
In this figure an hour is represented by a certain length, and the lengths representing hours are measured along a horizontal line. Each inch of height of the barometer is also represented by a certain length. At the points corresponding to the successive times perpendiculars are drawn, the lengths of which represent the heights of the barometer at the respective times. (In the figure the position of the horizontal line marked 29, represents the upper ends of heights of 29 inches.) The smooth curve drawn through the extremities of the perpendiculars is the *barometric curve* or *curve of barometric heights* for the period May 22 to May 29, 1899. This curve will give to most persons a clearer and more vivid idea of the range and variation of the height of the barometer during this period than a column of numbers of inches of heights is likely to give. If the scales used in representing the hours and the heights of the barometer were changed, then the curve would be somewhat altered, but its *general appearance would remain the same.*

The graph of a function. The graph of a function of x, say $f(x)$, is obtained in the following way: Take a horizontal line X_1OX; choose a point O, from which, distances representing the different values of x are measured along the line; measure positive values of x toward the right from O, and negative values toward the left. At particular points of X_1OX, at convenient distances apart, draw perpendiculars to represent the values of $f(x)$ at the respective points. Draw the perpendiculars upward from X_1OX when the values of $f(x)$ are positive, and downward when these values are negative. The smooth curve drawn through the extremities of these perpendiculars is the graph of $f(x)$. The nearer the perpendiculars

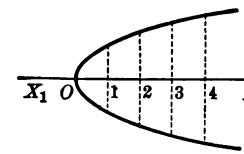
are to one another, the better is the graph. For example, the function $2x$ is represented (for certain values of x) by Fig. 72, the function $\frac{1}{2}x^2$, by Fig. 73, the function $\sqrt{x}$, by Fig. 74. The pupil is advised to construct these graphs by following the method just described above.



Graph of $2x$
FIG. 72.



Graph of $\frac{1}{2}x^2$
FIG. 73.



Graph of $\sqrt{x}$
FIG. 74.

EXS. Draw the graphs for $3x$, $\frac{1}{2}x$, $4x+5$, $4x-5$, $\frac{1}{4}x^2$, $\frac{1}{9}x^3$. $\sqrt{25-x^2}$.

NOTE. The notion of representing a function by a curve is the fundamental notion in algebraic geometry, or, as it is usually termed, analytic geometry. This geometry was invented, in the form in which it is now known, by the philosopher and mathematician, René Descartes (1596–1650), and first published by him in 1637. This article may be regarded as a short lesson in the subject.

82. Graphs of the trigonometric functions.

Graph of sin θ. In order to draw the graph of *sin θ* take distances, measured from O along the line X_1OX, to represent the number of radians in the angle θ. At points (not too far apart) on X_1OX draw perpendiculars to represent the sines of the angles corresponding to these points. The smooth curve drawn through the extremities of these perpendiculars will be the graph of the sine. Thus, for example, let *a radian* be represented by a *unit length*, and let *the ratio unity* be also represented by a *unit length*. Then (see Fig. 75) angle π (*i.e.* 180°) is represented by $OM_1 = 3\frac{1}{7}$. The perpendiculars at O and M_1 are zero, since $\sin 0 = 0$ and $\sin \pi = 0$. Erect perpendiculars equal to ···, sin 30°, sin 45°, sin 60°, sin 90°, ···, for instance, at the points corresponding to ···, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, ···, (*i.e.* ···, 30°, 45°, 60°, 90°, ···,) respectively. Do the

same at points between L and M_1, and draw the smooth curve OG_1M_1 through the extremities of the perpendiculars. The successive perpendiculars from π to 2π are the same in length as those from 0 to π, but negative. From 2π to 4π the values of the sine are repeated in the same order as from 0 to 2π. Hence, the graph of the sine can be obtained by merely successively reproducing the double undulation $OG_1M_1G_2M_2$, as indicated in Fig. 75. This is called the *curve of sines, sine curve,* or *sinusoid.*

NOTE 1. The unit circle (Art. 80) will be of service in drawing the graphs of the sine and the other trigonometric functions. For, if the scales for radians and ratios be those adopted above, then the *horizontal distances* from O will be equal to the *lengths of the arcs* (Art. 73, Note 2), and the *lengths of the perpendiculars* will be the *lengths of the lines in the line definitions* (Art. 79).

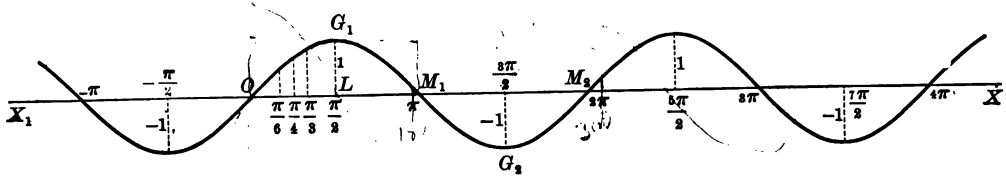
NOTE 2. If π radians (*i.e.* 180°) be represented by a length different from that adopted in Fig. 75, then the graph of the sine will differ somewhat from Fig. 75, but its main features will be the same as in that figure. Figures 76 and 77 show portions of the graph of $\sin\theta$ when π is represented on two other scales, while the $\sin\frac{\pi}{2}$ (*i.e.* 1) is represented by a unit length. Hence the curve of sines, or the sinusoid, may be defined as the curve in which horizontal distances measured on a certain line are proportional to an angle, and the perpendiculars to this line are proportional to its sine.

Ex. Draw the graphs for $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$.

Graph of cos θ. On using the same scales for radians and ratios as those adopted in Fig. 75, the graph of $\cos\theta$ takes the form shown in Fig. 78. It is the same as the graph of $\sin\theta$ in Fig. 75 would be if O and the other points in X_1OX were all moved a distance $\frac{1}{2}\pi$ toward the right. This might have been expected, since the sine of an angle is equal to the cosine of its complement. The values of the sine and the cosine alike range from $+1$ to -1.

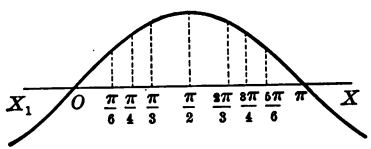
Graph of tan θ. On using the same scales for radians and ratios as have been adopted in Fig. 75, the graph of $\tan\theta$ takes the form shown in Fig. 79.

Graph of sec θ. On using the same scales for radians and ratios as have been adopted in Fig. 75, the graph of $\sec\theta$ takes the form shown in Fig. 80.



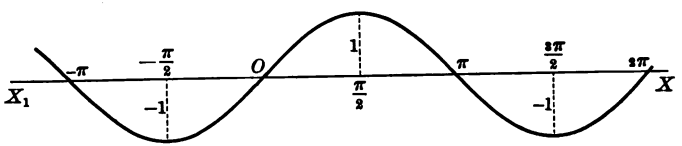
Graph of sin θ

FIG. 75.



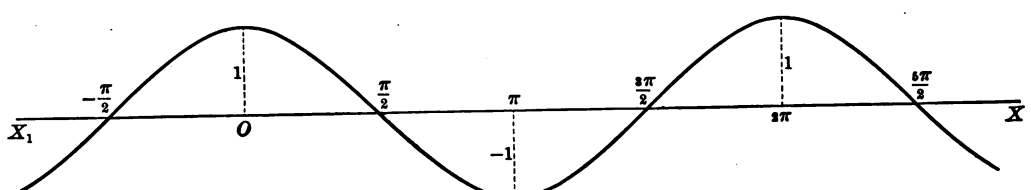
Graph of sin θ

FIG. 76.



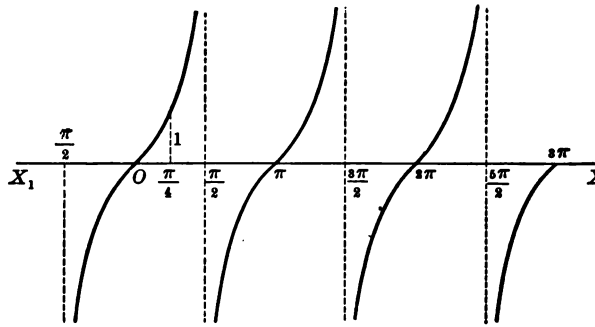
Graph of sin θ

FIG. 77.



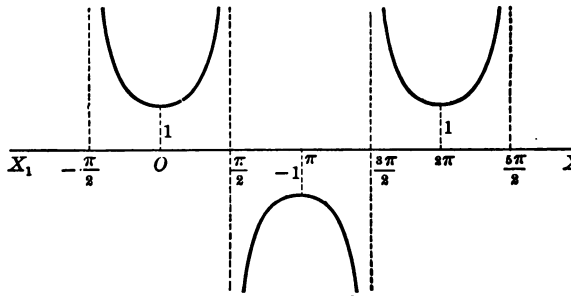
Graph of cos θ

FIG. 78.



Graph of $\tan\theta$

FIG. 79.



Graph of $\sec\theta$

FIG. 80.

EXAMPLES.

1. Draw the graph for $\cot\theta$, using the scales adopted in Fig. 75.

2. Draw the graph for $\operatorname{cosec}\theta$, using the scales adopted in Fig. 75.

3. Construct various graphs for $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$, by varying the scales used in representing radians and ratios.

4. Draw graphs for:

(*a*) $\sin x + \cos x$, (*b*) $\sin x - \cos x$, (*c*) $\sin 2x$, (*d*) $\cos 2x$.

83. Relations between the radian measure, the sine, and the tangent of an acute angle.

A. If θ be the radian measure of an acute angle, then $\sin\theta < \theta < \tan\theta$.

Let angle $AOP = \theta$, make the angle AOR equal to θ, and with any radius r describe the arc QBR about O as a centre. Draw the chord QR intersecting OB in M, and draw the tangents at Q and R. By geometry, arc QB = arc BR, QR is at right angles to OB, $MQ = MR$, the tangents at Q and R intersect at a point T on OA, $QT = RT$.

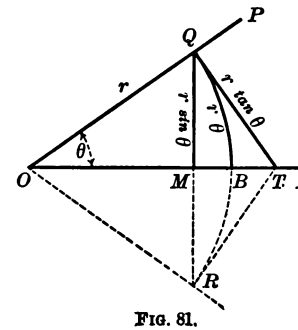


FIG. 81.

NOTE. If $r = 1$, that is, if QB is an arc of a unit circle, then the linear measures of MQ, BQ, TQ are equal to $\sin\theta$, θ, $\tan\theta$, respectively.

It can be easily shown *by mechanical means* that

$$MQ < BQ < TQ. \qquad (1)$$

For suppose that pegs are placed at Q, R, T, and that a string is drawn taut from Q to R; suppose that another string is drawn from Q to R, but constrained to lie on the circular arc QR, like a string stretched along the tire of a wheel. Also let a third string be drawn taut from R to Q, but passed over the peg at T. Then it is obvious that the first of the three strings is the shortest, and the third is the longest. The second string cannot be drawn away from the arc QBR without being stretched, and if peg T were removed, the string QTR would lie loosely on the arc QBR. Since $QMR < QBR < QTR$, then $MQ < BQ < TQ$; that is, $r\sin\theta < r\theta < r\tan\theta$. Hence, $\sin\theta < \theta < \tan\theta$.

The truth of A may be perceived from the following mathematical consideration. Draw the chord QB. Evidently,

area triangle $OQB <$ area sector $OQB <$ area triangle OQT.

That is, $\frac{1}{2} OB \cdot MQ < \frac{1}{2} OQ \cdot \text{arc } BQ < \frac{1}{2} OQ \cdot QT$;

or, $\frac{1}{2} r \cdot r \sin\theta < \frac{1}{2} r \cdot r\theta < \frac{1}{2} r \cdot r \tan\theta$.

Hence, $$\sin\theta < \theta < \tan\theta. \qquad (1)$$

B. When angle θ approaches zero, each of the ratios $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$, approaches unity as a limit. On dividing each of the members the inequality (1) by $\sin\theta$, there is obtained

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

But when θ approaches zero, $\cos\theta$ approaches unity as a limit (Art. 77). Hence, when θ approaches zero, $\frac{\theta}{\sin\theta}$ must also approach unity as a limit; that is, the limit of $\frac{\sin\theta}{\theta}$ is 1.

On dividing the members of (1) by $\tan\theta$, there is obtained

$$\cos\theta < \frac{\theta}{\tan\theta} < 1.$$

As before, when θ approaches zero, $\cos\theta$ approaches unity as a limit, and hence $\frac{\theta}{\tan\theta}$ approaches unity as a limit; *i.e.* the limit of $\frac{\tan\theta}{\theta}$ is 1. These results may be briefly expressed:

$$\underset{\theta=0}{\text{Limit}}\left(\frac{\sin\theta}{\theta}\right) = 1; \quad \underset{\theta=0}{\text{Limit}}\left(\frac{\tan\theta}{\theta}\right) = 1. \qquad (2)$$

These are two of the most important theorems in elementary trigonometry; they are frequently employed both in practical work and in pure mathematics.

A very important corollary to (2) is the following:

If θ be the radian measure of a very small angle, then θ can be used for $\sin\theta$ and $\tan\theta$ in calculations.

For instance, $\sin 10''$ to 12 places of decimals is .000048481368. This is also the radian measure of $10''$ to 12 places of decimals. The radian measures, sines, and tangents, of angles from 0° to 6°, agree in the first three places of decimals. For

$$\text{radian measure } 6° = (.10472) = .105; \ \sin 6° = (.10453) = .105;$$
$$\tan 6° = (.10510) = .105.$$

EXAMPLES.

1. Find the angle subtended by a man 6 ft. high at a distance of half a mile.

Here, $\theta = \tan\theta = \frac{6}{2640} = \frac{1}{440}$.

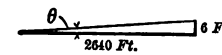


FIG. 82.

Now $\frac{1^r}{440} = \frac{1}{440} \times \frac{180^\circ}{\pi} = \frac{7 \times 180^\circ}{22 \times 440} = 7'\,48''.6.$

2. What must be the height of a tower, in order that it subtend an angle 1° at a distance of 4000 ft.?

$\frac{x}{4000} = \tan 1^\circ = \text{radian measure } 1^\circ = \frac{\pi}{180} = \frac{22}{7 \times 180}$.

FIG. 83.

$\therefore\ x = \frac{22 \times 4000}{7 \times 180} = 69.84$ ft.

3. Verify the statements made in Art. 11, Note 1. (Take $\pi = 3.14159265$.)

4. The moon's mean angular diameter as observed at the earth is $31'\,5''$, and its actual diameter is about 2160 miles. Find the mean distance of the moon. How many full moons would make a chaplet across the sky?

5. Taking the earth's equatorial radius as 3963 mi., find the angular semi-diameter of the earth as it would appear if observed from the moon. Compare the relative apparent sizes of the moon as seen from the earth, and the earth as seen from the moon.

6. The semi-diameter of the earth as seen from the sun is very nearly $8''.8$. (See Art. 11, Note 1.) What is the sun's distance from the earth, the radius of the earth being assumed as 4000 miles?

7. At least how many times farther away than the sun is the nearest fixed star α Centauri, at which the mean distance between the earth and sun (about 92,897,000 miles) subtends an angle something less than $1''$? How long, at least, will it take light to come from this star to the earth?

8. Find approximately the distance at which a coin an inch in diameter must be placed so as just to hide the moon, the latter's angular diameter being taken $31'\,5''$.

9. The inclination of a railway to a horizontal plane is $50'$. Find how many feet it rises in a mile.

10. Find the angle subtended by a circular target 4 ft. in diameter at a distance of 1000 yd.

11. Find the height of an object whose angle of elevation at a distance of 900 yd. is 1°.

12. Find the angle subtended by a pole 20 ft. high at a distance of a mile.

13. Exs. 5, 6, Art. 34 *b*.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter X. will be found at page* 195.

CHAPTER XI.

GENERAL VALUES. INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS.

84. General values. Articles 40–43 should be reviewed carefully before this chapter is taken up. It has been seen in these articles that all co-terminal angles have the same trigonometric ratios. It was also pointed out in Art. 43 that two sets of co-terminal angles, each set being infinite in number, correspond to any given ratio. For example, in Art. 42, Ex. 1, Fig. 38, any angle whose terminal line is either OP or OP_1 has a sine, $\frac{3}{4}$; in Ex. 2, Fig. 39, any angle whose terminal line is either OP or OP_1 has a tangent, $-\frac{3}{4}$. One of the objects of this chapter is to derive *expressions* or *formulas* that will include all angles which have the same sine, cosine, tangent, cotangent, secant, cosecant, respectively. These general expressions are sometimes called *general values.* The student is advised to deduce, after reading Art. 85, the general values for cosine, tangent, etc., without the help of the book.

85. General expression for all angles which have the same sine. Let s be the given value of the sine. It is required to find an expression that will represent and include every angle whose sine is s. All the angles whose sines are equal to s can be repre-

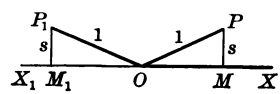


FIG. 84.

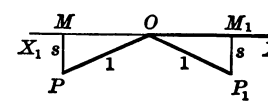


FIG. 85.

sented geometrically, as shown in Art. 42, and indicated in Figs. 84, 85. In Fig. 84, s is positive; in Fig. 85, s is negative.

Let XOP be the least positive angle whose sine is s. Let

$XOP = A$; then $XOP_1 = 180° - A$. Every angle whose terminal line is either OP or OP_1 has its sine equal to s. Now all angles having OP for a terminal line are obtained by adding all numbers of complete revolutions (positive and negative) to XOP. Hence, these angles are represented by

$$m \cdot 360° + A, \text{ i.e. } 2m \cdot 180° + A, \tag{1}$$

in which m is any positive or negative whole number.

Similarly, all angles having OP_1 for a terminal line are represented by

$$m \cdot 360° + (180° - A), \text{ i.e. } (2m + 1)\,180° - A. \tag{2}$$

An expression that will include both sets of angles, (1) and (2), will now be obtained. In the expression (1), the *coefficient* of 180° is *even*, and the *sign* of A is *positive;* in (2), the *coefficient* of 180° is *odd*, and the *sign* of A is *negative*. Hence, n being any positive or negative whole number, the expression

$$n \cdot 180° + (-1)^n A, \tag{3}$$

includes the angles in (1) and (2). This is, accordingly, the general expression for all the angles which have the same sine as A. If radian measure is used, and $XOP = \alpha$, then (3) takes the form

$$n\pi + (-1)^n \alpha. \tag{4}$$

The result may be thus expressed:

$$\sin A = \sin \{n \cdot 180° + (-1)^n A\}, \sin \alpha = \sin \{n\pi + (-1)^n \alpha\}. \tag{5}$$

Since $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$, *the general expression for all angles which have the same cosecant is the same as the general expression for all angles which have the same sine.*

EXAMPLES.

1. Find an expression to include all angles which have the same sine as 135°.

By (3), (4), the expression is $n \cdot 180° + (-1)^n 135°$, or $n\pi + (-1)^n \frac{3\pi}{4}$.

2. Find *the general value* of the angle whose sine is $+\frac{1}{\sqrt{2}}$. Give the four least positive angles which have sines equal to $+\frac{1}{\sqrt{2}}$.

The least angle whose sine is $+\frac{1}{\sqrt{2}}$ is 45°, *i.e.* $\frac{\pi}{4}$. Hence the general value is

$$n \cdot 180° + (-1)^n 45°, \text{ i.e. } n\pi + (-1)^n \frac{\pi}{4}.$$

To find the *four* least positive angles, put $n = 0, 1, 2, 3$, in this expression. This gives 45°, 135°, 405°, 495°, *i.e.* $\frac{\pi}{4}, \frac{3}{4}\pi, \frac{9}{4}\pi, \frac{11}{4}\pi$. These four angles can also be obtained by means of a figure.

3. Given that $\sin\theta = \frac{\sqrt{3}}{2}$: find the general value of θ, and find the *four* least positive values of θ.

4. As in Ex. 3 when $\sin\theta = -\frac{1}{2}$.

5. As in Ex. 3 when $\sin\theta = .95372$.

6. As in Ex. 3 when $\sin\theta = .39741$.

7. As in Ex. 3 when $\sin\theta = -.57833$.

86. General expression for all angles which have the same cosine. Let c be the given value of the cosine. It is required to find an expression to include every angle whose cosine is c. All the angles that have c for a cosine can be represented geometrically, as shown in Figs. 86, 87. In Fig. 86, c is positive; in Fig. 87, c is negative.

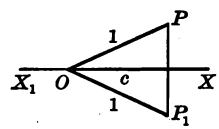


FIG. 86.

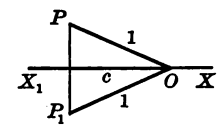


FIG. 87.

Let XOP be the least positive angle whose cosine is c, and let $XOP = A$ (in degree measure) $= a$ (in radian measure). Angle $XOP_1 = -A = -a$, also has its cosine equal to c. All angles whose terminal line is OP, have cosines equal to c. All these angles are included in

$$n \cdot 360° + A, \text{ i.e. } 2n\pi + a, \qquad (1)$$

in which n denotes any positive or negative whole number. Also, all angles whose terminal line is OP_1, have cosines equal to c. All these angles are included in

$$n \cdot 360° - A, \text{ i.e. } 2n\pi - a, \qquad (2)$$

α being as before. Both the expressions, (1), (2), are evidently included in

$$n \cdot 360° \pm A, \text{ or } 2n\pi \pm \alpha, \tag{3}$$

in which n is any positive or negative whole number. Hence (3) is the general expression for all angles which have the same cosine as A or α. The result may be thus expressed:

$$\cos A = \cos(n \cdot 360° \pm A); \ \cos\alpha = \cos(2n\pi \pm \alpha). \tag{4}$$

Since $\sec\theta = \dfrac{1}{\cos\theta}$, *the general expression for all angles which have the same secant is the same as the general expression for all angles which have the same cosine.*

EXAMPLES.

1. What is the general value of the angles which have the cosine, $-\frac{1}{2}$? Give the *three* least positive angles.

The least positive angle whose cosine is, $-\frac{1}{2}$, is 120°. Hence, the general value is, by (3), $n \cdot 360° \pm 120°$, *i.e.* $2n\pi \pm \frac{2}{3}\pi$. On putting $n = 0$ and 1, the three least positive angles are found to be 120°, 360° − 120°, or 240°, 360 + 120°, or 480°. These three angles may also be found by means of a figure.

2. Given that $\cos\theta = \dfrac{+\sqrt{3}}{2}$: find the general value of θ, and find the four least positive values of θ.

3. As in Ex. 2 when $\cos\theta = .99106$. **4.** As in Ex. 2 when $\cos\theta = .46690$.

5. As in Ex. 2 when $\cos\theta = -.72637$. **6.** As in Ex. 2 when $\cos\theta = -.40141$.

87. General expression for all angles which have the same tangent. Let t be the given value of the tangent. It is required to find an expression to include all angles which have the same tangent t.

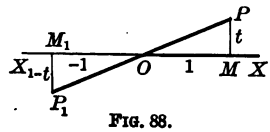


FIG. 88.

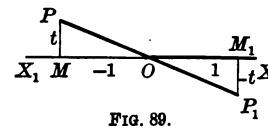


FIG. 89.

All the angles which have the same tangent t can be represented geometrically as in Figs. 88, 89. In Fig. 88, *the tangent* t is positive, in Fig. 89, it is negative.

Let $XOP = A$ (in degrees) $= a$ (in radians). Then

$$XOP_1 = 180° + A = \pi + a.$$

Each angle which has either OP or OP_1 for its terminal line, has its tangent equal to t. All the angles which have OP for a terminal line are included in the expression $m \cdot 360° + A$, that is, in

$$2m \cdot 180° + A, \text{ or } 2m\pi + a, \tag{1}$$

in which m denotes any positive or negative whole number.

All the angles which have OP_1 for a terminal line are included in the expression $m \cdot 360° + (180° + A)$, that is, in

$$(2m + 1)180° + A, \text{ or } (2m + 1)\pi + a. \tag{2}$$

Both these sets of angles, (1) and (2), are included in the expression

$$n \cdot 180° + A, \text{ or } n\pi + a, \tag{3}$$

in which n denotes any positive or negative whole number. Hence (3) is the general expression for all angles which have the same tangent as A or a. The result may be thus expressed:

$$\tan A = \tan(n \cdot 180° + A); \ \tan a = \tan(n\pi + a). \tag{4}$$

Since $\cot\theta = \dfrac{1}{\tan\theta}$, *the general expression for all angles which have the same cotangent is the same as the general expression for all angles which have the same tangent.*

EXAMPLES.

1. Find the general value of θ when $\tan\theta = 1$. The least positive angle whose tangent is 1, is $\frac{\pi}{4}$. Hence $\theta = n\pi + \frac{\pi}{4}$, in which n is any positive or negative whole number.

Find the general value of θ, and the four least positive values of θ when:

2. $\tan\theta = \sqrt{3}$. **3.** $\tan\theta = .36727$. **4.** $\tan\theta = 2.2998$.

5. $\tan\theta = .71769$. **6.** $\tan\theta = -.90040$. **7.** $\tan\theta = -2.6511$.

8. Find the general expression for all angles which have the same sine *and* cosine.

88. Inverse trigonometric functions. It has been seen that, on the one hand, the value of the sine depends on the value of the angle, and, on the other hand, the value of the angle depends on the value of the sine. If the angle is given, the sine can be determined; if the sine is given, the angle can be expressed. Hence, on the one hand, *the sine is a function of the angle*, and, on the other hand, *the angle is a function of the sine.* The latter function is said to be **the inverse** function of the former. The same holds in the case of each of the other trigonometric functions. Inverse functions are usually denoted by the symbol described below.

The two statements: *the sine of the angle θ is m,* (1)

θ is the angle whose sine is m, (2)

are briefly expressed: $\sin\theta = m,$ (3)

$\theta = \sin^{-1} m.$ (4)

The symbols $\sin^{-1} m$, $\cos^{-1} m$, $\tan^{-1} m$, $\cdots$, are called *inverse trigonometric functions*, or *anti-trigonometric functions*, or *inverse circular functions.* The symbol "$sin^{-1} m$" is read, "angle whose sine is m," "anti-sine of m," "inverse sine of m," "sine minus one m." It should be carefully remembered that *here, -1 is not an algebraical exponent*, but is merely part of a mathematical symbol; $\sin^{-1} m$ does not denote $(\sin m)^{-1}$, that is, $\frac{1}{\sin m}$; $\sin^{-1} m$ denotes each and every angle whose sine is m. *The trigonometric functions are pure numbers; the inverse circular functions are angles,* and are denoted by *the number of degrees or radians in these angles.* For instance, if $\theta = \frac{\pi}{4}$ in (3), then $m = +\frac{1}{\sqrt{2}}$;

if $m = +\frac{1}{\sqrt{2}}$ in (4), then $\theta = \sin^{-1}\left(\frac{1}{+\sqrt{2}}\right) = n\pi + (-1)^n \frac{\pi}{4}$

$= n \cdot 180° + (-1)^n 45°,$

in which n is any whole number. This example illustrates what has already been noted in Arts. 42, 43, 78, namely:

For a given value of the angle θ, $\sin\theta$ or m has *a single definite value.*

For a given value of the sine m, $\sin^{-1} m$ or θ has *an infinite number of values.*

The same is the case with each of the other inverse trigonometric functions. Thus the **trigonometric functions are single-valued,** and the **inverse circular functions are multiple-valued.**

For example, if $\cos\theta = \frac{\sqrt{3}}{2}$, then $\theta = \cos^{-1}\frac{\sqrt{3}}{2} = 2n\pi \pm \frac{\pi}{6}$ (Ex. 2, Art. 86); if $\theta = \tan^{-1} 1$, then $\theta = n\pi + \frac{\pi}{4}$ (Ex. 1, Art. 87), in which n denotes any whole number. The *smallest numerical value* of an inverse trigonometric function is called the **principal value** of the inverse function. For instance, the principal value of $\sin^{-1}\frac{1}{2}$ is 30°, of $\tan^{-1}(-1)$ is $-45°$, of $\cos^{-1}(-\frac{1}{2})$ is 120°, of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is $-60°$.

NOTE 1. In some books the symbols *arc sin x*, *arc cos x*, *arc tan x*, ···, are used for inverse trigonometric functions. These symbols are read, "arc sine x," ···. The derivation of these names is apparent from Art. 79.

NOTE 2. *Algebraic.* If y is a function of x, say $f(x)$, then x also depends on y, and hence, is some function of y. This function of y is called the *inverse function of* $f(x)$ *or* y, and is usually denoted by $f^{-1}(y)$. For instance, if $y = f(x) = x^2$, then $x = f^{-1}(y) = \pm\sqrt{y}$.

It will be observed in this simple example that, while the function of x has a single value, the inverse function has two values. In other words, y is a *single-valued* function of x, and x is a *two-valued* function of y. As shown above, if $y = \sin x$, then $x = \sin^{-1} y$; y is a single-valued function of x, but x is a multiple-valued function of y.

It appears from Notes 1, 2, that the English notation for inverse trigonometric functions avoids the old geometrical conceptions of trigonometric functions, and is also more general in character. The inverse trigonometric functions are frequently met in calculus and applied mathematics.

89. Sum and difference of two anti-tangents. Exercises on inverse functions.

Find $\tan^{-1} m + \tan^{-1} n$, and $\tan^{-1} m - \tan^{-1} n$.

Let $x = \tan^{-1} m$, and $y = \tan^{-1} n$.

Then $\tan x = m$, $\tan y = n$.

Now $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ (Art. 51) $= \dfrac{m+n}{1-mn}$.

$\therefore x + y = \tan^{-1}\dfrac{m+n}{1-mn}$; *i.e.* $\boldsymbol{\tan^{-1} m + \tan^{-1} n = \tan^{-1}\dfrac{m+n}{1-mn}}$. (1)

In a similar manner it can be shown that

$$\boldsymbol{\tan^{-1} m - \tan^{-1} n = \tan^{-1}\frac{m-n}{1+mn}}. \qquad (2)$$

EXAMPLES.

1. Find $\tan^{-1}2 \pm \tan^{-1}\frac{1}{3}$. (Compare Ex. 1, Art. 51.)

$$\tan^{-1}2 + \tan^{-1}\tfrac{1}{3} = \tan^{-1}\frac{2+\frac{1}{3}}{1-2\cdot\frac{1}{3}} = \tan^{-1}7 = n\cdot 180° + 81°\,52'\,11''.5.$$

$$\tan^{-1}2 - \tan^{-1}\tfrac{1}{3} = \tan^{-1}\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}} = \tan^{-1}1 = n\pi + \frac{\pi}{4}.$$

By the tables, taking acute angles only, $\tan^{-1}2 = 63°\,26'\,4''.3$, $\tan^{-1}\frac{1}{3} = 18°\,26'\,6''$, the sum is $81°\,52'\,10''.3$, and the difference is $44°\,59'\,58''.3$. The slight discrepancy between the results obtained by the two methods is due to the fact that the angles found by the tables are only approximately correct.

In the following examples test or verify the result in the manner shown in Ex. 1.

2. Find $\tan^{-1}7 \pm \tan^{-1}3$.

3. Find $\tan^{-1}2 + \tan^{-1}.5$.

4. Find $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

5. Find $\tan^{-1}3 + \tan^{-1}2 + \tan^{-1}.6$.

(Suggestion. Find $\tan^{-1}3 + \tan^{-1}2$, then combine the result with $\tan^{-1}.6$.)

6. Find $2\tan^{-1}1.5$, $2\tan^{-1}3$, $2\tan^{-1}2$, $3\tan^{-1}.2$.

7. Show that $2\tan^{-1}m = \tan^{-1}\frac{2m}{1-m^2}$. Show that $2\theta = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$.

8. Show that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$ when the angles are between 0° and 90°.

9. Find $\sin(\sin^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{3})$ when the angles are between 0° and 90°.

10. When the angles are between 0° and 90°, show that:

(*a*) $\sin(\sin^{-1}m \pm \sin^{-1}n) = m\sqrt{1-n^2} \pm n\sqrt{1-m^2}$.

(Suggestion. Let $x = \sin^{-1}m$, $y = \sin^{-1}n$.)

(*b*) $\cos(\sin^{-1}m \pm \sin^{-1}n) = \sqrt{1-n^2}\,\sqrt{1-m^2} \mp mn$.

(*c*) $\sin(\sin^{-1}m \pm \cos^{-1}n) = mn \pm \sqrt{1-m^2}\,\sqrt{1-n^2}$.

(*d*) $\cos(\sin^{-1}m \pm \cos^{-1}n) = n\sqrt{1-m^2} \mp m\sqrt{1-n^2}$.

11. Find $\sin(\sin^{-1}\frac{2}{3} + \sin^{-1}\frac{4}{7})$, $\cos(\sin^{-1}\frac{1}{7} - \cos^{-1}\frac{1}{8})$,
$\sin(\cos^{-1}\frac{2}{5} - \cos^{-1}\frac{3}{10})$, $\sin(\tan^{-1}4 - \cos^{-1}\frac{2}{3})$, $\tan(\sec^{-1}3 - \sin^{-1}\frac{1}{2})$,
(*a*) when the angles are between 0° and 90°, (*b*) when this restriction is not imposed.

12. Two lines, AB, AC, intersect a horizontal line at B, C, making angles whose tangents are $\frac{1}{3}$, $\frac{2}{7}$. Find the angle BAC.

13. Two lines, LM, LN, make angles whose tangents are $\frac{1}{3}$, 2, with a horizontal line. Find the angle MLN.

90. Trigonometric equations. Trigonometric equations have appeared in many of the preceding articles. When an angle, θ say, is the unknown quantity in a trigonometric equation, *the complete solution is the general value of θ which satisfies the equation.* For example, if a be an angle whose sine is s, then the solution of the equation,

$$\sin\theta = s, \text{ that is, of } \theta = \sin^{-1}s,$$

is $\theta = n\pi + (-1)^n a$, n being any integer.

EXAMPLES.

(See the definition of principal value in Art. 88.)

1. Solve the equation $\cos\theta = \frac{\sqrt{3}}{2}$.

The principal value of θ is $\frac{\pi}{6}$. Hence the complete solution is $\theta = 2n\pi \pm \frac{\pi}{6}$, n being any integer. (See Ex. 2, Art. 86.)

2. Solve the equation $\theta = \tan^{-1}1$.

The principal value is $\frac{\pi}{4}$. $\therefore\ \theta = n\pi + \frac{\pi}{4}$. (See Ex. 1, Art. 87.)

3. Solve the equation $\sin x \cos x = -\frac{1}{4}\sqrt{3}$.

$\therefore\ \sin x\sqrt{1-\sin^2 x} = -\frac{1}{4}\sqrt{3}$. $\therefore\ \sin^2 x(1-\sin^2 x) = \frac{3}{16}$.

$\therefore\ \sin^4 x - \sin^2 x + \frac{3}{16} = 0$. $\therefore\ (\sin^2 x - \frac{3}{4})(\sin^2 x - \frac{1}{4}) = 0$.

$\therefore\ \sin^2 x = \frac{3}{4}$; $\sin^2 x = \frac{1}{4}$.

Whence (*a*) $\sin x = \pm\sqrt{\frac{3}{4}}$; (*b*) $\sin x = \pm\frac{1}{2}$.

The given equation shows that $\sin x$ and $\cos x$ have opposite algebraic signs. Hence, x can only be in the second and fourth quadrants.

$\therefore$ In (*a*), $x = 120°, 300°$, etc., its general value is $n\cdot 180 - 60°$, where n is any positive integer.

In (*b*), $x = 150°, 330°$, etc.; its general value is $n\cdot 180° - 30°$, n being any positive integer.

4. Solve the equation $\sin 5\theta + \sin\theta = \sin 3\theta$.

$\therefore\ 2\sin 3\theta\cos 2\theta = \sin 3\theta$. $\therefore\ \sin 3\theta(2\cos 2\theta - 1) = 0$.

$\therefore$ (*a*) $\sin 3\theta = 0$, (*b*) $2\cos 2\theta - 1 = 0$.

From (*a*), $3\theta = 0°, 180°$, etc.; the general value of 3θ is $n\pi$ (n being any integer).

$\therefore\ \theta = 0°, 60°$, etc.; the general value of θ is $\frac{n\pi}{3}$.

From (*b*), $\cos 2\theta = \frac{1}{2}$. $\therefore\ 2\theta = \pm 60°$, etc.; its general value is $2n\pi \pm \frac{\pi}{3}$.

$\therefore\ \theta = \pm 30°$, etc.; its general value is $n\pi \pm \frac{\pi}{6}$.

Find solutions of these equations:

5. $3(\sec^2\theta + \cot^2\theta) = 13$.
6. $\cot\theta - \tan\theta = 2$.
7. $\sec x + \tan x = 2$.
8. $\sec^2 x + \tan x = 7$.
9. $\sec^2 x - \tan x = 3$.
10. $\cos\theta - \cos 7\theta = \sin 4\theta$.
11. $2\sin x + 5\cos x = 2$.
12. $\sin 2\theta + \sin 4\theta = \sqrt{2}\cdot\cos\theta$.
13. $4\sin\theta\cos 2\theta = 1$.
14. $\tan^4 A - 4\tan^2 A + 3 = 0$.
15. $3(\tan^2\theta + \cot^2\theta) = 10$.
16. $\cos^{-1}x - \sin^{-1}x = \cos^{-1}x\sqrt{3}$.

N.B. *Questions and exercises suitable for practice and review on the subject-matter of Chapter XI. will be fouud at pages* 197–199.

CHAPTER XII.

MISCELLANEOUS THEOREMS AND EXERCISES.

91. Chapters II.–VIII. were devoted to the oldest and the simplest application of trigonometry; namely, the measurement of triangles. Angles and the trigonometric functions connected with angles were more fully discussed in Chapters IX.–XI. This chapter does not introduce any new principles. Most of its articles may be regarded as exercises on the relations shown in Chapters II.–VIII., and more especially on the properties announced in Arts. 44, 50–52. The articles just mentioned should be reviewed. Some of the results in the exercises in this chapter are useful and important; but *the student should direct attention mainly to the methods whereby the results are obtained,* so that he can proceed quickly and confidently to the solutions of similar exercises. These solutions require a ready and an accurate knowledge of (that is, an intelligent familiarity with) the formulas deduced in the earlier chapters. It is on this account, perhaps, that such exercises are regarded with favor by examiners.

92. Functions of twice an angle. Functions of half an angle.

Relations (5)–(8), Art. 50, (3), Art. 51, give the sine, cosine, and tangent of twice an angle in terms of the functions of the angle. On rearranging (7), (8), Art. 50, there is obtained,

$$\sin A = \sqrt{\frac{1-\cos 2A}{2}}, \quad \cos A = \sqrt{\frac{1+\cos 2A}{2}}; \tag{1}$$

whence,

$$\tan A = \frac{\sin A}{\cos A} = \sqrt{\frac{1-\cos 2A}{1+\cos 2A}}. \tag{2}$$

On putting $\frac{1}{2}x$ for A, these relations take the forms

$$(a)\ \sin \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}, \qquad (b)\ \cos \tfrac{1}{2}x = \sqrt{\frac{1+\cos x}{2}},$$

$$(c)\ \tan \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{1+\cos x}}. \tag{3}$$

In (1), (2), (3), angles A and x denote any angles.

EXERCISES.

1. Account for the ambiguity of the radicals in (1), (2), (3).

2. Find $\sin 45°$, given that $\cos 90° = 0$.

From (3) a, $\quad \sin 45° = \sqrt{\dfrac{1 - \cos 90°}{2}} = \dfrac{1}{\sqrt{2}}$.

3. Find $\sin 22° 30'$, $\cos 22° 30'$, $\tan 22° 30'$ by means of (1), (2). Compare the values with those given in the tables.

93. Functions of three times an angle. Functions of an angle in terms of functions of one-third the angle.

To express tan 3 A in terms of tan A. Let A denote any angle.

$$\tan 3A = \tan(2A + A)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan^2 A}{1 - \tan^2 A}}. \quad \text{(Art. 51.)}$$

$$\therefore \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}. \qquad (1)$$

On putting x for $3A$, (1) becomes

$$\tan x = \frac{3\tan \frac{1}{3}x - \tan^3 \frac{1}{3}x}{1 - 3\tan^2 \frac{1}{3}x}$$

To express sin 3 A in terms of sin A.

$$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A \quad \text{[Art. 50, (1)]}$$
$$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A.$$

$$\therefore \sin 3A = 3\sin A - 4\sin^3 A. \qquad (2)$$

In a similar way, $\cos 3A$ can be expressed in terms of $\cos A$.

$$\cos 3A = 4\cos^3 A - 3\cos A. \qquad (3)$$

EXERCISES.

1. Derive formula (3).

2. On substituting x for $3A$, write (2), (3).

3. Express formulas (1), (2), (3), and the results of Ex. 2, *in words*.

4. Assuming the value of $\sin 30°$, calculate $\sin 90°$.

5. From $\cos 30°$, derive $\cos 90°$; from $\tan 30°$, derive $\tan 90°$.

6. Derive $\sin 180°$, $\cos 180°$, $\tan 180°$, from $\sin 60°$, $\cos 60°$, $\tan 60°$, respectively.

7. Derive $\sin 75°$, $\cos 75°$, $\tan 75°$, from $\sin 25°$, $\cos 25°$, $\tan 25°$, respectively, as given in the tables.

8. Derive $\sin 37° 30'$, $\cos 37° 30'$, $\tan 37° 30'$, from the ratios of $75°$.

94. Functions of the sum of three angles.

$$\left.\begin{aligned}\tan(A+B+C) &= \tan(\overline{A+B}+C) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}\\ &= \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\cdot\tan C}\\ &= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.\end{aligned}\right\}\quad(1)$$

COR. 1. If $A=B=C$, (1) reduces to (1), Art. 93.

COR. 2. If $A+B+C=180°$, then $\tan(A+B+C)=0$, and, accordingly, the numerator of (1) is equal to zero. Hence, if A, B, C, are the three angles of a triangle,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C. \qquad (2)$$

COR. 3. If $A+B+C=90°$, then $\tan(A+B+C)=\infty$, and, accordingly, the denominator of (1) is equal to zero. Hence,

$$\tan A\tan B+\tan B\tan C+\tan C\tan A=1, \text{ when } A+B+C=90°. \qquad (3)$$

EXERCISES.

1. Show that $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$.

If $A+B+C=180°$, the first member is zero. Division of the second member by $\cos A\cos B\cos C$ will give relation (2) above.

2. Show that $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C$. What does this become when

$$A+B+C=180°?$$

If $A+B+C=90°$, the first member is zero. Division of the second member by $\cos A\cos B\cos C$ will give relation (3) above.

3. If $A+B+C=180°$, prove that

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Since $A+B+C=180°$, $\quad\frac{A+B}{2}=90°-\frac{C}{2}$.

$$\cos A+\cos B+\cos C=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos C \qquad \text{[Art. 52 (7)]}$$

$$=2\sin\frac{C}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2} \qquad \text{[Art. 50 (7)]}$$

$$=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)$$

$$=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$=1+2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2} \qquad \text{[Art. 52 (8)]}$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

4. If $A+B+C=180°$, prove that

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

5. If $A+B+C=180°$, prove that

$$\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

6. Also, that $\sin(A+B)\sin(B+C)=\sin A\sin C$.

7. Also, that $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$.

8. If $A+B+C=90°$, show that

$$\sin 2A+\sin 2B-\sin 2C=4\sin A\sin B\cos C.$$

9. Find $\tan 4A$, $\tan 5A$, $\tan 6A$, $\tan 7A$, in terms of $\tan A$.

95. Identities. In the following exercises it is required that the first member be changed into the second member. When it is difficult to do this, help is sometimes afforded by taking some steps in changing the second member into the first. The direct steps to be taken from the first member to the second may be indicated by this means. No general directions can be given concerning the making of these transformations. The two following suggestions, however, are frequently useful:

(*a*) Since $\sin^2 A+\cos^2 A=1$, unity can be substituted for the first expression, and the first expression can be substituted for unity.

(*b*) The change of $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, into their values in terms of the sine and cosine, is sometimes helpful.

The examples in Art. 52 belong to this class.

EXERCISES.

1. Show that $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.

$$\frac{1-\cos 2A}{1+\cos 2A} = \frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)} = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A.$$

2. Show that $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$.

$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{\frac{1}{2}(1-\cos 2A)}{\frac{1}{2}(1+\cos 2A)} = \frac{1-\cos 2A}{1+\cos 2A}$$

NOTE. The fact that $\tan^2 A = \dfrac{\sin^2 A}{\cos^2 A}$, suggests that the numerator in Ex. 1 be expressed in terms of the sine, and the denominator in terms of the cosine. In Ex. 2, the plan of transformation is more obvious.

Prove the following identities:

3. $\dfrac{\sec^2 B}{2-\sec^2 B} = \sec 2B$.

4. $1 - 2\sin^2(45° - A) = \sin 2A$.

5. $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$.

6. $1 + \cot 2\theta \cot\theta = \operatorname{cosec} 2\theta \cot\theta$.

7. $4\sin A \sin(60° + A)\sin(60° - A) = \sin 3A$.

8. $\cos 5\theta = \cos(3\theta + 2\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

9. $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

10. $\tan(45° + A) - \tan(45° - A) = 2\tan 2A$.

11. $\cos^4 B - \sin^4 B = \cos 2B$.

12. $\dfrac{\sin 3A - \cos 3A}{\sin A + \cos A} = 2\sin 2A - 1$.

13. $\dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$.

14. $4(\cos^6 x + \sin^6 x) = 1 + 3\cos^2 2x$.

15. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

16. $\cos 4A = 1 - 8\cos^2 A + 8\cos^4 A$.

96. For an acute angle of θ radians, $\cos\theta > 1 - \dfrac{\theta^2}{2}$, $\sin\theta > \theta - \dfrac{\theta^3}{4}$.

By Art. 50, (7), $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$; by Art. 83, $\sin\dfrac{\theta}{2} < \dfrac{\theta}{2}$.

Hence, $\cos\theta > 1 - 2\left(\dfrac{\theta}{2}\right)^2$, *i.e.* $\cos\theta > 1 - \dfrac{\theta^2}{2}$.

By Art. 50, (5),

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = 2\tan\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right).$$

But by Art. 83, $\tan\frac{\theta}{2} > \frac{\theta}{2}$, and $\sin\frac{\theta}{2} < \frac{\theta}{2}$.

Hence $\sin\theta > \frac{2\theta}{2}\left\{1 - \left(\frac{\theta}{2}\right)^2\right\}$; *i.e.* $\sin\theta > \theta - \frac{\theta^3}{4}$.

97. One method of computing the trigonometric functions. A method of computing the trigonometric functions of angles which are in an arithmetic progression having the common difference D'', will now be shown.

$\sin(n+1)D'' + \sin(n-1)D'' = 2\sin nD''\cos D''$. [Art. 52, (5).]

$\therefore \quad \sin(n+1)D'' = 2\sin nD''\cos D'' - \sin(n-1)D''$. (1)

Also $\cos D'' = \sqrt{1 - \sin^2 D''}$.

Hence, if the sines of the angles D'', $2D''$, $3D''$, up to nD'' be known, then $\sin(n+1)D''$ can be computed by formula (1). The other functions can be derived from the sine.

The functions for angles from 0° to 45° will serve for the angles from 45° to 90°, since the ratio of an angle is the co-ratio of its complement. When the functions have been computed for angles up to 30°, the computations for angles greater than 30° can be made more easily. For, if A is an angle less than 30°,

$$\sin(30° + A) + \sin(30° - A) = 2\sin 30°\cos A = \cos A.$$

$$\therefore \quad \sin(30° + A) = \cos A - \sin(30° - A). \tag{2}$$

Similarly, $\cos(30° + A) = \cos(30° - A) - \sin A$. (3)

In formula (1) suppose that $D'' = 10''$, and let its radian measure be denoted by θ.

Then $\sin 10'' < \theta, > \theta - \frac{\theta^3}{4}$. [Arts. 83, 96.]

Since $180° = \pi$,

$$10'' = \frac{10\pi}{180 \times 60 \times 60} = \frac{3.1415926535}{64800} = .000048481368 \cdots \text{ radians.}$$

$\therefore \sin 10'' < .00004848\cdots, > [.00004848\cdots - \frac{1}{4}(.00004848\cdots)^3]$.

Hence, to 12 places of decimals, $\sin 10'' = .000048481368$.

From this, $\sin 20''$ can be found by (1); then $\sin 30''$, then $\sin 40''$, and so on.

The functions of several angles can be found independently of the method just shown. Formulas involving these angles, and

Euler's and Legendre's *verification formulas*, may be used to test the accuracy of the tables. The latter formulas are (see Exs. 7–10, Ch. XII.),

$$\sin(36°+A)-\sin(36°-A)-\sin(72°+A)+\sin(72°-A)=\sin A, \quad (4)$$

$$\cos(36°+A)+\cos(36°-A)-\cos(72°+A)-\cos(72°-A)=\cos A. \quad (5)$$

EXERCISES.

1. Test the tables of natural sines and cosines by means of formulas (4), (5), taking A equal to 4°, 10°, 15°, and other values. **2.** Assuming the functions of 1° as known, calculate the sines of 2°, 3°, 4°, 5°, 6°, by formula (1). **3.** By means of formulas (2), (3), calculate the sines and cosines of 33°, 37°, 41°, 47°, 53°, 67°, and other angles.

98. Trigonometry defined. Branches of trigonometry. Before concluding this text-book it may be well to indicate to the student the relation of the part of trigonometry treated in the preceding pages to the subject as a whole, and also to try to give him a little idea of another branch of trigonometry; namely, analytical trigonometry.

In Chapters II.–IX., plane angles, the solution of plane triangles, and applications connected therewith were discussed. This is what is usually known as *plane trigonometry*. The study of solid angles, the solution of spherical triangles, and the associated practical applications, constitute *spherical trigonometry*. These branches of mathematics are founded on geometrical considerations, and may be looked upon as applications of algebra to geometry. Pure mathematics is sometimes regarded as consisting of two great branches; namely, *geometry* and *analysis*. Analysis includes algebra, infinitesimal calculus, and other subjects which employ the symbols, rules, and methods of algebra, and do not rest upon conceptions of space. (Geometrical ideas may be used in analysis, however, for the sake of exposition and illustration, and, on the other hand, algebra may be employed in expounding the principles of geometry.) Since the eighteenth century, trigonometry has also been treated as a *branch of analysis*.*

* The meaning of the word "analysis" thus used in mathematics, should not be confounded with the ordinary meaning of the word, or with the meaning attached to the term "analysis" in logic.

Analytical (or algebraical) *trigonometry* treats of the general relations of angles and their trigonometric functions without any reference to measurement. It discusses, among other things, the development of exponential and logarithmic series, the connections between trigonometric and exponential functions, the expansions of an angle and its trigonometric functions into infinite series, the calculation of π, the summation of series, and the factorization of certain algebraic expressions. The properties stated in formulas, (1)–(3) Art. 44, (1)–(8) Art. 50, (1)–(8) Art. 52, (1)–(3) Art. 93, are analytical properties, and *can be derived without the aid of geometry.* Analytical trigonometry includes hyperbolic trigonometry; that is, the treatment of what are called the hyperbolic functions.

While the trigonometric functions may be defined and discussed on a geometrical basis, as done in this book (and this is the easiest way for beginners), it may be stated that they can also be defined and their properties deduced on a purely algebraic basis. It is beyond the scope of this work to show this, but the student may obtain a little light on the subject by reading Notes A and D. It may be stated further, that, under certain restrictions, some of the most important theorems and properties found in analytical trigonometry can be derived easily in an elementary course in the infinitesimal calculus. It has been pointed out that the trigonometric functions can be defined in a purely geometrical manner, and in a purely algebraic manner; they can also be given definitions depending on the infinitesimal calculus, and their properties deduced therefrom. Finally, it may be said that trigonometry is merely a brief chapter in the modern Theory of Functions, and may be defined as the science of *singly periodic functions* (see Art. 78). For a treatment of trigonometry, either as a part of algebra, or, as "an elementary illustration of the application of the Theory of Functions," see Lock, *Higher Trigonometry;* Loney (Part II.), *Analytical Trigonometry;* W. E. Johnson, *Treatise on Trigonometry,* Chaps. XII.–XXII.; Casey, *A Treatise on Plane Trigonometry;* Levett and Davison, *Elements of Plane Trigonometry* (Parts II., III., Real Algebraical Quantity, Complex Quantity); Hayward, *Vector Algebra and Trigonometry;* Hobson, *A Treatise on Plane Trigonometry;* Chrystal, *Algebra,* Part I., Chap. XII.; Part II., Preface, and Chaps. XXIX., XXX.

3. The sine and cosine of the sum of two angles. [Supplementary to Art. 46.]

Let the construction be made as indicated in Art. 46. Then

$$\sin(A+B) = \frac{\text{proj. } OP \text{ on } OY}{OP} = \frac{\text{proj. } OQ \text{ on } OY}{OP} + \frac{\text{proj. } QP \text{ on } OY}{OP}.$$

[Theorem in (2).]

$$= \frac{\text{proj. } OQ \text{ on } OY}{OQ} \cdot \frac{OQ}{OP} + \frac{\text{proj. } QP \text{ on } OY}{QP} \cdot \frac{QP}{OP}$$

$$= \sin A \cos B + \sin VQP \sin B$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\cos(A+B) = \frac{\text{proj. } OP \text{ on } OX}{OP} = \frac{\text{proj. } OQ \text{ on } OX}{OP} + \frac{\text{proj. } QP \text{ on } OX}{OP}$$

$$= \frac{\text{proj. } OQ \text{ on } OX}{OQ} \cdot \frac{OQ}{OP} + \frac{\text{proj. } QP \text{ on } OX}{QP} \cdot \frac{QP}{OP}$$

$$= \cos A \cos B + \cos VQP \cdot \sin B$$

$$= \cos A \cos B - \sin A \sin B.$$

In the projection proof of the addition formulas for the sine and cosine, A and B can have any magnitudes, positive or negative. The formulas for $\sin(A-B)$, $\cos(A-B)$, can also be derived by substituting $-B$ for $+B$ in the addition formulas.

NOTE C.

[Supplementary to Arts. 9, 72.]

ON THE LENGTH AND AREA OF A CIRCLE.

1. The main purpose of this note is to outline a method of approximating to the value of π; that is, to the ratio of the length of a circle to its diameter. This method depends only on elementary geometry.* There are simpler and more expeditious methods of finding π, but they require a greater knowledge of mathematics than beginners in trigonometry generally possess.

By the methods of elementary geometry, as shown in the texts of Euclid and others, regular polygons of 3, 4, 5, 6, 15 sides can be inscribed in, and circumscribed about a given circle. Moreover, inscribed and circumscribing regular polygons of 2, 4, 8, 16, ···, times each of those numbers of sides can also be constructed by successively bisecting the arcs subtended by the sides, and joining the consecutive points of division. This process can evidently be

* A section on the mensuration of the circle is given in many geometries. Reference may be made to the geometries of Beman and Smith (Ginn & Co.), Gore (Longmans, Green, & Co.), Phillips and Fisher (Harpers), and others.

carried on until the inscribed and circumscribing polygons have an infinitely great number of sides; that is, regular polygons of 3.2^n, 4.2^n, 5.2^n, 15.2^n sides, n being any positive integer, can be inscribed in, or circumscribed about, a given circle.

2. Outline of a proof of the theorem that the lengths of circles are proportional to their diameters.

(*a*) The length of a circle is greater than the perimeter of an inscribed polygon, and is less than the perimeter of a circumscribing polygon of any finite number of sides.

(*b*) As the number of sides of a regular polygon inscribed in, or circumscribed about, a circle is increased, the length of the perimeter of the polygon approaches nearer and nearer to the length of the circle. In other words, by increasing the number of sides, the difference between the length of the perimeter of the polygon and the length of the circle may be made as small as one please, and this difference approaches zero when the number of sides approaches infinity.

(*c*) Let any two circles be taken, and let the radii be R, r. Let AB be a side of a regular polygon of n sides inscribed in the circle having centre O

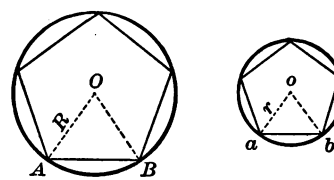


FIG. 92.

and radius R, and let ab be the side of a regular polygon of n sides inscribed in the circle having centre o and radius r. Let P denote the perimeter of the first polygon, p that of the second; let C denote the length of the first circle, and c that of the second. Then

$$P = C - D, \quad p = c - d,$$

where D, d may each be made smaller than any assignable quantity by making the number of sides, n, infinitely great.

The polygons are similar, since they are regular and have the same number of sides. Hence, by geometry,

$$\frac{P}{p} = \frac{OA}{oa} = \frac{R}{r};$$

that is,

$$\frac{C - D}{c - d} = \frac{R}{r}.$$

From this,

$$rC - rD = Rc - Rd;$$

whence,

$$rC - Rc = rD - Rd.$$

Now, let n become infinitely great. Then the second member becomes smaller than any assignable quantity, since r, R, each remains finite, and d, D, each approaches zero. Hence, when n is infinitely great,

$$rC - Rc = 0. \tag{1}$$

From (1), $$\frac{C}{c} = \frac{R}{r}, \text{ and } \frac{C}{R} = \frac{c}{r}. \tag{2}$$

The first of equations (2) may be expressed in words: lengths of circles are to one another as their radii. According to the second equation, the length of the first circle is to its radius as the length of the second circle is to *its* radius. But these are any two circles. Hence, the ratio of the length of a circle to its radius, and, consequently, to its diameter, is constant. The ratio of the length of a circle to its diameter, which ratio is denoted by π, will now be approximately determined. [See Arts. 9 (*b*), (*c*), 72.]

3. The formulas used in this determination of π are deduced in problems ***A***, ***B***, that follow:

A. *Given the radius of a regular inscribed polygon, to compute the side of a similar circumscribing polygon.*

Let AB be the side of the inscribed polygon, and $OC = R$, the radius of the circle; let LM be a side of the similar circumscribing polygon. Let LM be obtained by producing OA, OB, to intersect the tangent drawn at C, the middle point of the arc AB.

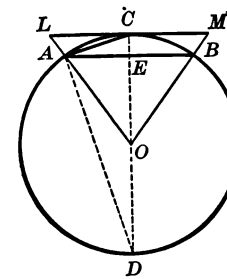


FIG. 93.

The triangles LCO, AEO, are similar. Hence,

$$\frac{LC}{AE} = \frac{OC}{OE}.$$

$$\therefore LC = \frac{OC \times AE}{OE} = \frac{R \times AE}{OE}.$$

$$\therefore LM = \frac{R \times AB}{OE}.$$

In the right-angled triangle OAE,

$$OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{R^2 - \frac{\overline{AB}^2}{4}} = \frac{1}{2}\sqrt{4R^2 - \overline{AB}^2}.$$

$$\therefore LM = \frac{2R \times AB}{\sqrt{4R^2 - \overline{AB}^2}}. \tag{1}$$

B. *Given the radius and the side of a regular inscribed polygon, to compute the side of the regular inscribed polygon of double the number of sides.*

In Fig. 93, let AB be the side of a regular inscribed polygon of n sides. Draw AC; then AC is the side of a regular inscribed polygon of double the number of sides, namely, $2n$ sides. It is required to express AC in terms of the radius R and the side AB. Produce CO to D and draw DA. The triangles ACD, ACE, are similar, since the angles DAC, AEC, are equal,

both being right angles, and the angle ACE is common to both triangles Hence,

$$CD : AC = AC : CE.$$

$$\therefore \overline{AC}^2 = CD \cdot CE = CD(CO - EO) = 2R(R - EO) = R(2R - 2EO).$$

But $$EO = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{R^2 - \frac{\overline{AB}^2}{4}} = \frac{1}{2}\sqrt{4R^2 - \overline{AB}^2}.$$

$$\therefore \overline{AC}^2 = R(2R - \sqrt{4R^2 - \overline{AB}^2}).$$

$$\therefore AC = \sqrt{R(2R - \sqrt{4R^2 - \overline{AB}^2})} \qquad (2)$$

4. To determine approximately the ratio of the circumference of a circle to its diameter. If the radius is 1, the length of the circle is 2π, and the length of the semicircle is π. Hence the length of the semi-perimeter of each inscribed and circumscribing regular polygon, is an approximate value of π, and approaches nearer and nearer to π, the greater the number of sides in the polygon. The side of the *inscribed square* of the circle of radius 1 is $\sqrt{2}$; its semi-perimeter is 2.8284271. Successive applications of (2), Art. 3, give the sides of the inscribed polygons of 8, 16, 32, ··· sides, and successive applications of (1) give the sides of the similar circumscribing polygons. The successive semi-perimeters are obtained by taking one-half the product of the length of a side and the number of sides in the polygons. The results of the computation are given in the following table. The table also gives the results when the initial polygon taken, is the *inscribed hexagon*. *The figures in bold type show the approximations.*

Lengths of semi-perimeters of regular inscribed and circumscribing polygons of circle of radius = 1.

Number of Sides.	Inscribed.	Circumscribing.	Number of Sides.	Inscribed.	Circumscribing.
4	2.8284271	4.0000000	6	**3**	**3**.4641016
8	**3**.0614675	**3**.3137085	12	**3**.1058285	**3**.2153903
16	**3**.1214452	**3**.1825979	24	**3**.1326286	**3**.1596599
32	**3**.1365485	**3**.1517249	48	**3**.1393502	**3**.1460862
64	**3**.1403312	**3**.1441184	96	**3**.1410319	**3**.1427146
128	**3**.1412773	**3**.1422236	192	**3**.1414524	**3**.1418730
256	**3**.1415138	**3**.1417504	384	**3**.1415576	**3**.1416627
512	**3**.1415729	**3**.1416321	768	**3**.1415838	**3**.1416101
1024	**3**.1415877	**3**.1416025	1536	**3**.1415904	**3**.1415970
2048	**3**.1415914	**3**.1415951			
4096	**3**.1415923	**3**.1415933			
8192	**3**.1415926	**3**.1415928			

5. Area of a circle. Area of a circular sector. The area of a circumscribing polygon of a circle is equal to one-half the product of the lengths of the perimeter and the radius. When the number of sides of the polygon increases indefinitely, the perimeter of the polygon approaches the length of the circle as its limit, and the area of the polygon approaches the area contained by the circle as its limit. Hence,

$$\text{area of circle} = \tfrac{1}{2} \text{ length of circle} \times \text{length of radius};$$

i.e. $$\textbf{area of circle} = \tfrac{1}{2} \times 2\pi R \times R = \pi R^2.$$

Since the area of a sector of a circle has the same ratio to the area of the circle that the arc of the sector has to the length of the circle,

$$\text{area of circular sector} = \tfrac{1}{2} \text{ length of arc} \times \text{length of radius}.$$

Hence, if θ is the radian measure of the angle of the sector,

$$\textbf{area sector} = \tfrac{1}{2} R\theta \times R = \frac{1}{2} R^2\theta.$$

[For example, see Art. 73, Ex. 17.]

6. Historical Note. The problem to find a square whose area is equal to that of a given circle, which is commonly known as "squaring the circle," or "the quadrature of the circle," has long been of interest to mathematicians and others. Since the area of a circle is one-half the radius by the length of the circle, and the ratio of the length of the circle to the diameter is a constant, it follows that "squaring the circle" comes to determining this ratio.

The ancient peoples used 3 as the value of π; see 1 Kings vii. 23, 2 Chron. iv. 2. Ahmes used 3.1604; Archimedes showed, by the method described above, and by successively inscribing and circumscribing regular polygons of 6, 12, 24, 48, 96 sides, that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$. Ptolemy used $3\frac{17}{120}$, and Aryabhatta, 3.1416. Adrian of Metz, in 1527, by using polygons up to 1536 sides, showed that the ratio is between $\frac{377}{120}$ and $\frac{333}{106}$. By taking the mean of the numerators for a new numerator, and the mean of the denominators for a new denominator, he obtained the value $\frac{355}{113}$, which is correct to six places of decimals. In 1579, Vieta by using polygons of 32,316 (*i.e.* 6×2^{16}) sides, got the value of π correctly to ten places. His method is not the same as that of Archimedes. (Professor Newcomb has remarked that the value of π to *ten* places of decimals would give the circumference of the earth correctly to within a fraction of an inch, if the diameter were accurately known.) In 1593, Adrian Romanus of Louvain computed π to 15 places of decimals and Ludolph van Ceulen (d. 1610), a German residing in Holland, calculated it to 35 places. Hence π is often referred to in Germany as "the Ludolphian number."

The discoveries of trigonometric series (see Note A) made the work of computers easier and more mechanical. In 1699, Abraham Sharp (1651–1742),

found π to 71 places, and in 1706, John Machin, died 1751, professor of astronomy at Gresham College, London, extended the value to 100 places. Fautet de Lagny (1660–1734) carried it, in 1719, to 127 places; Baron Georg Vega (1756–1802), in 1794, to 136 places; Z. Dase of Vienna, in 1844, to 200 places; William Rutherford (1798?–1871), Royal Military Academy, Woolwich, in 1853, to 440 places; Richter, in 1854, to 500 places; and W. Shanks, in 1873, to 707 places of decimals. The laborious calculations of the "π-computers" have neither theoretical nor practical value.

About 1761 Lambert showed that π is *incommensurable*, and in 1882, F. Lindemann, in Freiburg, showed that it is *transcendental*, that is, it cannot be a root of any algebraic equation with integral coefficients. See article, "Squaring the circle," Ency. Brit., 9th edition.

N.B. The ratio π is often calculated approximately by means of Gregory's series (discovered in 1670) and certain identities, namely,

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots \text{ to } \infty, \quad \frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239},$$

$$\frac{\pi}{4} = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{5} + \tan^{-1}\tfrac{1}{8}, \qquad \frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99},$$

in which the principal values of the inverse tangents are taken.

The student is advised to verify these identities, and to find an approximate value of π by means of Gregory's series; also to verify the remark made above concerning the circumference of the earth. Also to show that the method of Ahmes (see Note A) for finding the area of a circle is equivalent to taking $\pi = 3.1604\ldots$.

NOTE D.

DE MOIVRE'S THEOREM, AND OTHER RESULTS IN ANALYTICAL TRIGONOMETRY.

[In what follows i denotes $\sqrt{-1}$.]

1. De Moivre's Theorem. For all values of n, positive and negative, integral and fractional,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(a) *When n is a positive integer.*

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2). \qquad (1)$$

On multiplying each member of (1) by $\cos\theta_3 + i\sin\theta_3$, there is obtained,

$$\begin{aligned}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\\ &= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)\\ &= \{\cos(\theta_1+\theta_2)\cos\theta_3 - \sin(\theta_1+\theta_2)\sin\theta_3\}\\ &\qquad + i\{\sin(\theta_1+\theta_2)\cos\theta_3 + \cos(\theta_1+\theta_2)\sin\theta_3\}\\ &= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3).\end{aligned}$$

In a similar way, the product of four or more such factors can be found. Thus, for n factors,

$$\begin{aligned}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)\\ &= \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n) \qquad (2)\end{aligned}$$

If $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$, then (2) becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad (3)$$

(*b*) *When n is a negative integer, say,* $-m$.

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-m} &= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}\\ &= \frac{1}{\cos m\theta + i\sin m\theta}\cdot\frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\ &= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta. \qquad (4)\end{aligned}$$

(*c*) In (3) let $n\theta = \phi$; then $\theta = \frac{\phi}{n}$, and (3) becomes

$$\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)^n = \cos\phi + i\sin\phi.$$

On transposing and taking the nth root of each member of this equation, there is obtained

$$(\cos\phi + i\sin\phi)^{\frac{1}{n}} = \cos\frac{\phi}{n} + i\sin\frac{\phi}{n}. \qquad (5)$$

(The second member of (5) is *one* of the n roots of the first member.)

(*d*) *When n is a fraction,* $\frac{p}{q}$.

$$\begin{aligned}(\cos\theta + i\sin\theta)^{\frac{p}{q}} &= [(\cos\theta + i\sin\theta)^p]^{\frac{1}{q}} = (\cos p\theta + i\sin p\theta)^{\frac{1}{q}}\\ &= \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta. \qquad (6)\end{aligned}$$

(The second member of (6) is *one* of the q roots of the first member.)

For all the roots of the first members of (5), (6), see one of the works referred to in Art. 98. For a geometrical representation of the factors considered above and the results (1)–(6), and for a proof of these results, see Hobson, *Plane Trigonometry*, Chap. XIII.; Chrystal, *Algebra*, Part I., Chap. XII.

2. Following are some of the theorems proved in analytical trigonometry, which will be met by those who read only a little farther in mathematics:

$$\cos x = 1 - \frac{x^2}{2\,|} + \frac{x^4}{4\,|} - \cdots \text{ to } \infty. \tag{1}$$

$$\sin x = x - \frac{x^3}{3\,|} + \frac{x^5}{5\,|} - \cdots \text{ to } \infty. \tag{2}$$

Expansions (1) and (2) were first shown by Newton in 1669.

$$\text{If } e^x \text{ denote the series } 1 + x + \frac{x^2}{2\,|} + \frac{x^3}{3\,|} + \cdots \text{ to } \infty, \tag{3}$$

$$\text{then} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \ \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{4}$$

Formulas (4) were first given by Euler. The expansions (1), (2), (3), are also derived in works on the differential calculus. They are convergent for all finite values of x. Either (1), (2), or (3), (4), may be taken as *definitions* of the sine and cosine.

Hyperbolic functions. The hyperbolic sine and cosine of x, denoted by *sinh* x, *cosh* x, may be defined in either one of the following ways, namely,

$$\left.\begin{aligned} \cosh x &= 1 + \frac{x^2}{2\,|} + \frac{x^4}{4\,|} + \cdots \text{ to } \infty \\ \sinh x &= x + \frac{x^3}{3\,|} + \frac{x^5}{5\,|} + \cdots \text{ to } \infty \end{aligned}\right\} \quad \cdots\cdots \tag{5}$$

$$\text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2}. \tag{6}$$

A geometrical definition may also be given to the hyperbolic functions. In this definition, they are related to the hyperbola in a manner analogous to a way in which the trigonometric (circular) functions are related to a circle. It may be said that the formulas or definitions (1)–(6) may be applied to all numbers x, real, pure imaginary, or complex; *i.e.* quantities of the form $a + b\sqrt{-1}$. (When x is real in (1)–(4), it denotes the radian measure of the angle.) See Chrystal, *Algebra*, Part II., Chap. XXIX.*

EXERCISES.

1. Substitute 1 for x in the series (3), and thus deduce 2.71828 as an approximate value of e.

2. (*a*) Write the series for e^{ix} and e^{-ix} by substituting ix and $-ix$ for x in (3).

(*b*) Then find the value of $\cos x$ in (4), and compare the result with (1).

(*c*) Then find the value of $\sin x$ in (4), and compare the result with (2).

3. Using formulas (4), show that $\cos^2 x + \sin^2 x = 1$.

4. Substitute ix for x in (1) and (2), and compare the results with (5). Substitute ix for x in (4), and compare the results with (6). Each substitution shows that $\cos(ix) = \cosh x$, and $\sin(ix) = i \sinh x$.

5. Show by means of the formulas (1)–(4) that the cosine of an angle of magnitude zero is unity, and that the sine of such an angle is zero.

6. By means of (1), (2), find approximate values of $\sin 10°$, $\cos 10°$, $\sin 15°$, $\cos 15°$, $\sin 20°$, $\cos 20°$, $\sin 30°$, $\cos 30°$. [First, express the angles in radian measure.]

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* Also see McMahon, *Hyperbolic Functions*, (Merriam and Woodward, *Higher Mathematics*, Chap. IV., pp. 107–168).

QUESTIONS AND EXERCISES FOR PRACTICE AND REVIEW.

It is not intended that all these exercises be worked by any one person, or by any one class. It is advisable to consider only a few of them on the completion of each chapter, and to use them chiefly in the general reviews. In each set there are a number of direct questions on the principles and theorems explained in the corresponding chapter; these questions will enable the pupil to examine himself concerning his knowledge of the text. Teachers will often do well to take examples from other sources.

CHAPTER I.

1. (*a*) Define and illustrate *logarithms*, *characteristic*, *mantissa*. (*b*) What is meant by the base of a system of logarithms? (*c*) What is meant by a system of logarithms? (*d*) Show that in all systems $\log 1 = 0$, and that the logarithms of all proper fractions are negative.

2. What are the advantages gained by the use of logarithms calculated to the base 10? Show that the characteristic of any logarithm to the base 10 may be found by inspection.

3. What are the logarithms to base 3, of 81, $\frac{1}{81}$, $\sqrt[4]{729}$?

4. Find, by using logarithms, the values of the following quantities: (*a*) $\sqrt{375}$, $\sqrt{37.5}$, $\sqrt{3.75}$, $\sqrt{.375}$, $\sqrt{.0375}$; (*b*) $\sqrt[3]{12.5}$, $\sqrt[3]{1.25}$, $\sqrt[3]{.125}$, $\sqrt[3]{.0125}$, $\sqrt[3]{.00125}$; (*c*) $\sqrt[5]{784}$, $\sqrt[7]{93}$, $\sqrt[9]{834}$; (*d*) $(19)^{\frac{4}{5}}$, $(212)^{\frac{2}{3}}$, $(31.7)^{\frac{3}{5}}$; (*e*) 4^{-7}, $5^{-\frac{1}{3}}$, $(67)^{-\frac{2}{3}}$.

5. The following calculations are to be made at first without the use of logarithmic tables. The results may then be checked by comparing them with the values obtained by means of the tables. (*a*) Given $\log 3 = .477121$, find $\log \{(2.7)^3 \times (81)^{\frac{4}{5}} \div (90)^{\frac{3}{4}}\}$. (*b*) Given $\log 5 = .69897$, find $\log 200$, $\log .025$, $\log \sqrt[5]{62.5}$. (*c*) Given $\log 2 = .30103$, find the logarithms of 5, $\frac{1}{125}$, $\sqrt[4]{.005}$. (*d*) Given $\log 2 = .3010$ and $\log 3 = .4771$, find $\log \frac{1}{24}$, $\log .25$.

log 16.2, log $\sqrt{\frac{2}{3}}$. (*e*) Given log 2 = .3010, log 3 = .4771, log 5 = .6990, log 7 = .8451, find the logarithms of $\frac{2}{15}$, 175, .0054, $(12)^{\frac{2}{3}}$, $\sqrt[4]{35}$. (*f*) Given log 8 = .903, log 9 = .954, find the logarithms of 2, 3, 12, 500, .075.

6. Find by logarithms the values of the following quantities:

(*a*) $372.48 \times (\frac{11}{151})^{60}$, $(.006)^5 \div 125$; (*b*) $3487 \times (.00345)^{\frac{1}{2}} \div (-83)^{\frac{1}{3}}$; (*c*) $\sqrt[5]{\frac{3}{7}} \div \sqrt[7]{\frac{9}{35}}$; (*d*) $\left(\frac{345.4 \times 958.3}{23.4 \times 317.9}\right)^{\frac{1}{2}}$, $\left(\frac{417.9 \times 813.1}{964.7 \times 313.2}\right)^{\frac{1}{2}}$; (*e*) $(23\frac{3}{7})^{\frac{2}{3}} \times (41\frac{2}{3})^{\frac{3}{5}} \div (7.93)^{3.5}$.

7. Find an approximate value of x in each of the following equations: (*a*) $x^2 = 237$, (*b*) $x^3 = 17$, (*c*) $x^{-2} = 17$, (*d*) $x^{-3} = 4\frac{1}{2}$, (*e*) $2^{2x} = 9$, (*f*) $3^{5x} = 197$, (*g*) $3^x = 32$, (*h*) $(25)^{3-2x} = 2^{x+3}$, (*i*) $\log(x^2) + \log(2x) + 1 = 0$.

8. If the logarithm of 27 is $-\frac{3}{2}$, what is the base?

CHAPTER II.

1. If on a map a square inch represents 10 acres, how many yards are represented by the diagonal of a square inch?

2. Explain the English and French methods of measuring angles, and show how, when the measure of an angle according to either method is known, its measure according to the other may be found. Express 100° in grades. (See Note 2, Art. 11.)

3. If A is an acute angle, show that $\tan A$ is greater than $\sin A$.

4. By aid of an equilateral triangle find the numerical values of the six trigonometric ratios of 60° and 30°. Find the numerical values of the ratios of 45°.

5. Show that (*a*) $\sqrt{\frac{\sin 45° - \sin 30°}{\sin 45° + \sin 30°}} = \sec 45° - \tan 45°$,

(*b*) $\frac{1 + \cot 60°}{1 - \cot 60°} = \left\{\frac{1 + \cos 30°}{1 - \cos 30°}\right\}^{\frac{1}{2}}$,

(*c*) $\tan^2 60° - 2\tan^2 45° = \cot^2 30° - 2\sin^2 30° - \frac{3}{4}\operatorname{cosec}^2 45°$.

6. The sine of an angle defined as a ratio being less than unity, explain why the tabular logarithms of the sines of angles are expressed with whole numbers as characteristics. Given log tan 18° = 9.51178, show what the tabular logarithm of cot 18° must be.

7. (*a*) Given log 2 = .30103, log 3 = .47712, find log sin 60° and log tan 30°. (*b*) Given log 5 = .69897, find the logarithmic sine of 30°, and the logarithmic cosine of 45°.

8. Compute the trigonometric ratios of A in a right triangle ABC ($C = 90°$), when $b = \frac{4}{5}c$.

9. Construct the following right-angled triangles: (*a*) ABC, in which, $C = 90°$, $c = 5$, $\cot A = \frac{7}{5}$; (*b*) when one of the legs is 3, and the sine of the adjacent acute angle is $\frac{2}{5}$; (*c*) hypotenuse 4, and sine of one of the acute

angles $\frac{3}{5}$; (*d*) $C = 90°$, $\sin A = \frac{3}{5}$, $b = 7$; (*e*) $C = 90°$, $\operatorname{cosec} A = \frac{5}{4}$, $b = 10$; write the values of $\sin A$, $\cos A$, $\tan A$; (*f*) $C = 90°$, $\cos A = \frac{3}{5}$, $a = 9$.

10. In the triangle ABC, $C = 90°$, $\tan B = 1\frac{1}{4}$. If $AB = 510$ ft., find AC.

11. In ABC, $C = 90°$, $BC = 10$ ft., $\tan B = 1.05$; find the other sides.

12. The string of a kite is 250 ft. in length. How high is the kite above the ground when the string, supposed stretched quite tight, makes with the ground an angle whose tangent is $\frac{24}{7}$?

13. ABC is an isosceles triangle, right-angled at C; D is the middle point of AC. Prove that DB divides the angle B into two parts whose cotangents are as 2 : 3.

14. (*a*) Given L. cos $20° = 9.97$ and L. cot $20° = 10.44$; find *each* of the other logarithmic ratios of 20°. (*b*) Given L. sin $40° = 9.808$, L. tan $40° = 9.924$; find log cot 40°, log cos 40°, log sec 40°, log cosec 40°.

15. If $\tan\theta = \dfrac{2\sqrt{ab}\sin\dfrac{C}{2}}{a-b}$, find θ when $a = 5$, $b = 2$, $C = 120°$.

16. Calculate $\sin^2 23° \times \sqrt{27.268} \div 2\cos^2 48°$.

17. Find x in the equations: (*a*) $x \sin 74° = 235 \tan 37° \cos 63°$,
(*b*) $x^2 \cos 39° = 47.5 \sin^2 46° \sec^2 64°$.

18. Solve $(\sin 8° + \cos 8°)^{2x} = 2 \sin 16° (\tan 32°)^x$.

CHAPTER III.

1. State what parts of a right plane triangle must be given that it may be constructed, and show how a right triangle may be solved, in each of the four possible cases.

2. Derive the formulas for computing B, a, and c of a right triangle when $C = 90°$, and A and b are given. Also find a formula that shall include only the required parts.

3. In ABC, $C = 90°$, $b = 22$ ft., and $\sin A = .42$. Find a, c, $\sin B$, and the area.

4. In ABC, $C = 90°$, $\cos A = \frac{7}{10}$, $c = 40$ ft. Find the values of $\cos B$, $\cot B$, a, b, and the area.

5. Solve the following right-angled triangles by (1) making an off-hand estimate, (2) measuring on a drawing made to scale, (3) computing without logarithms (four-place tables), (4) computing with logarithms. Check the results by computation. If a solution is impossible, explain why it is so. (Each triangle is denoted by ABC, and $C = 90°$.) (i.) $a = 45$, $b = 62$; (ii.) $a = 685$, $B = 34° 47' 25''$; (iii.) $c = 560$, $a = 310$; (iv.) $c = 327$, $b = 450$; (v.) $c = 520$, $A = 36° 40' 20''$; (vi.) $b = 720$, $B = 61° 24' 30''$; (vii.) $c = 425$, $B = 32° 45' 35''$; (viii.) $a = 11524$, $b = 35976$; (ix.) $a = 67213$, $b = 75324$; (x.) $c = 35421$, $b = 23462$.

6. Two sides of a triangle are as 5 : 9, and the included angle is a right angle. Find the other angles.

7. Find the acute angles of a right-angled triangle whose hypotenuse is six times as long as the perpendicular let fall upon it from the opposite angle.

CHAPTER IV.

1. (*a*) Derive the formula for the area of a right triangle in terms of (i.) an angle and its opposite side, (ii.) an angle and its adjacent side. (*b*) One side of a triangle is seven times another, and the included angle is a right angle. Find the other angles.

2. Show how an isosceles triangle may be divided into right triangles, and how it may be solved by aid of these right triangles when the following elements are given: (*a*) Base and vertical angle, (*b*) base and side, (*c*) side and vertical angle, (*d*) base and perpendicular from vertex on the base. Discuss any other possible cases.

3. Solve the isosceles triangles (*a*) whose base is 126 ft., and vertical angle is 127°; (*b*) whose base and perpendicular on it from the vertex are each 721.34 yd.

4. (*a*) Find the area of a regular octagon the side of which is 26 yd. (*b*) Find the side of a regular pentagon inscribed in a circle whose radius is 43 ft. (*c*) If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.

5. (*a*) At 120 ft. distance, and on a level with the foot of a steeple, the angle of elevation of the top is 62° 27′; find the height. (*b*) From a cliff 330 ft. high the angle of depression of a boat at sea is 40° 35′ 25″; how far is the boat from the foot of the cliff?

6. When the altitude of the sun is 30° the length of the shadow cast by Bunker Hill monument is 381 ft. What is the height of the monument?

7. The angles of depression from the top of a tower 48.6 ft. high to two points, on a level with its base and in line with the tower, are 45° and 30° respectively. Find the distances of each point from the other and from the top of the tower.

8. A pole 40 ft. high is erected at the intersection of the diagonals of a square courtyard. When the sun's altitude is 43° 40′, the shadow just reaches a corner of the yard. Find the length of the side of the square.

9. (*a*) When the altitude of the sun was 67° 30′ 45″ the length of the shadow of a perpendicular pole was 73.4 ft. Find the length of the shadow when the sun's altitude is 35°. (*b*) The shadow of a tower is observed to be half the known length of the tower, and some time after to be equal to the full length. How much will the sun have gone down in the interval?

10. A flagstaff which leans to the east is found to cast shadows of 198 ft. and 202 ft., when the sun is due east and west respectively, and his altitude is 7°. Find the length of the flagstaff and its inclination to the vertical.

11. What angle will a flagstaff 24 ft. high, on the top of a tower 200 ft. high, subtend to an observer on the same level with the foot of the base, and 100 yds. distant from it?

12. Looking out of a window with his eye at the height of 15 ft. above the roadway, an observer finds that the angle of elevation of the top of a telegraph post is 17° 18′ 35″, and that the angle of depression of the foot of the post is 8° 32′ 15″. Calculate the height of the telegraph post and its distance from the observer.

13. A man in a balloon, when it is one mile high, finds the angle of depression of an object on the level ground to be 35° 20′, then after ascending vertically and uniformly for 20 min., he finds the angle of depression of the same object to be 55° 40′. Find the rate of ascent of the balloon in miles per hour.

14. A man observes the elevation of a mountain top to be 15°, and after walking 3 mi. directly toward it on level ground, the elevation is 18°. Find his distance from the mountain.

15. From a boat the angle of elevation of the highest and lowest points of a flagstaff, 30 ft. high, on the edge of a cliff are observed to be 46° 12′ and 44° 13′. Determine the height of the cliff and its distance.

16. The angles of elevation of the top of a tower, observed at two points in the horizontal plane through the base of the tower, are $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{5}{12}$; the points of observation are 240 ft. apart, and lie in a direct line from the base. Find the height of the tower.

17. A person standing due south of a lighthouse observes that his shadow cast by the light at the top is 24 ft. long; on walking 100 yd. due east he finds his shadow to be 30 ft. Supposing him to be 6 ft. high, find the height of the light from the ground.

18. An observer is 384 yd. due south of a point from which a balloon ascended; he measures a horizontal base 112 yds. east, and at the other extremity finds the angle of elevation to be 60° 15′. Find the height of the balloon.

19. A surveyor starts from A and runs 766 yd. due east to B, thence 622 yd. N. 20° 30′ E. to C, thence 850 yd. N. 41° 45′ W. to D, thence S. 42° 35′ W. to E. Find the distance and bearing of A from E, and determine the area of the field $ABCDE$.

20. A surveyor runs 253 yd. N.E. by E., thence N. by E. 212 yd., thence W.N.W. 156 yd., thence S.W. by S. 210 yd., thence to the starting-point. Find the bearing and distance of the starting-point from the last station, and determine the area of the field which the surveyor has gone around.

CHAPTER V.

1. Define and illustrate *angle*, *negative angle*, *complement of an angle*, *supplement of an angle*, *quadrant*, *angle in the third quadrant*.

2. Define and illustrate the six *trigonometric ratios*. Find the greatest and least values that each of them can have. Arrange in tabular form the algebraic signs of the trigonometric ratios of an angle in each quadrant.

3. Explain how the trigonometric ratios of an angle of any magnitude, positive or negative, can be found, (*a*) by means of tables which give these ratios for angles up to 90° only, (*b*) by means of tables which give these ratios for angles up to 45° only.

4. Prove that if two angles have the same sine, and also any of the other five trigonometric ratios (with one exception) the same, they will differ by a multiple of 360°.

5. State and prove the chief relations which exist between the trigonometric ratios of any angle A.

6. Express the trigonometric ratios of $90° - A$, $90° + A$, $180° - A$, $180° + A$, $270° - A$, $270° + A$, $360° - A$, $-A$, in terms of the trigonometric ratios of A.

7. Name *three* pairs of trigonometric ratios such that the product of each pair shall equal 1; *one* pair, the sum of whose squares shall equal 1; *two* pairs, the difference of whose squares shall equal 1.

8. Compare the trigonometric ratios of any angle (*a*) with those of its complement, (*b*) with those of its supplement.

9. Prove that $\sin A = \cos A \tan A$; $\sec^2 A = 1 + \tan^2 A$; $\cot A = \operatorname{cosec} A \cos A$; $\sin^2 A + \cos^2 A = 1$; $\sin\theta = \tan\theta : \sqrt{1 + \tan^2\theta}$; $\cos x = \sqrt{\operatorname{cosec}^2 x - 1} : \operatorname{cosec} x$.

10. (*a*) Express the following trigonometric ratios in terms of trigonometric ratios of positive angles not greater than 45° : sin 237°, cos (− 410°), tan 2000°, cot (− 137°), sec 445°, cosec (− 650°), sin 185°, tan 267°, sec 345°, cos 87°, cot (− 19°); (*b*) by means of the tables give the numerical values of these ratios.

11. Find, without the use of trigonometric tables, the numerical values of cos 1410°, tan (− 1260°), cosec (− 1710°), tan 225°, cot 1035°, cosec 210°, cos 1500°, sin 1665°, tan (− 1665°), all the trigonometric ratios of − 1125° and 930°.

12. Construct the angles: (*a*) whose secant is 3, (*b*) whose tangent is $\sqrt{2} + 1$, (*c*) whose cotangent is $\frac{4}{3}$. Find the other ratios of these angles.

13. (*a*) Find sin A, cot A, when $\cos A = -\frac{5}{13}$, and $A < 180°$. (*b*) Find the other ratios of A and x when $\cot A = \frac{8}{15}$ and $\cos x = -\frac{4}{5}$. (*c*) Find the other ratios of A when $\cos A = -\frac{1}{2}$, and A lies between 540° and 630°.

(*d*) Find the trigonometric ratios of $180° + \theta$ and $270° - \theta$, given $\tan\theta = \frac{1}{2}$. (*e*) Given $\sec x = -\frac{4}{3}$, and x in the third quadrant; find the value of $\dfrac{\sin x + \tan x}{\cos x + \cot x}$.

14. Do Ex. 9, Art. 18, A being any angle. Explain the ambiguities in the algebraic signs. If A is an angle in the third quadrant, express $\cos A$, $\tan A$, $\cot A$, $\sec A$, $\operatorname{cosec} A$ in terms of $\sin A$.

15. (*a*) If $\sec A = n \tan A$, find the other ratios of A. (*b*) If $2\sec\theta = \tan\alpha + \cot\alpha$, find $\tan\theta$ and $\operatorname{cosec}\theta$. (*c*) Solve $x^3 \cot 108° = 128 \sin 72° \cos 18°$.

16. Prove the identities: $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$; $\cos^4 A - \sin^4 A = 1 - 2\sin^2 A$; $\sin x(\cot x + 2)(2\cot x + 1) = 2\operatorname{cosec} x + 5\cos x$; $\sec^2 B - \cos^2 B = \cos^2 B \tan^2 B + \sin^2 B \sec^2 B$; $\cos^6 A + \sin^6 A = 1 - 3\cos^2 A \sin^2 A$; $\cos^6 x + 2\cos^4 x \sin^2 x + \cos^2 x \sin^4 x + \sin^2 x = 1$.

17. (*a*) Find the value of x not greater than two right angles which will satisfy the equation $4\sqrt{3}\cot x = 7\operatorname{cosec} x - 4\sin x$. (*b*) Likewise, in the case of the equation $\sin x + \cos x \cot x = 2$. (*c*) Likewise, in $\tan^4 x - 4\tan^2 x + 3 = 0$. (*d*) If $1 + \sin^2\theta = 3\sin\theta\cos\theta$, find $\tan\theta$. (*e*) Find the least positive value of A that satisfies the equation $2\sqrt{3}\cos^2 A = \sin A$. (*f*) Find all the angles between 0° and 500° which satisfy the equation $4\sin^2\theta = 3$. (*g*) If $2\cos A + \sec A = 3$, what is the value of A? (*h*) Find A when $\tan^2 A + \operatorname{cosec}^2 A = 3$.

CHAPTER VI.

1. (*a*) Write the values of $\cos(A + B)$, $\cos(A - B)$, $\sin(A + B)$, $\sin(A - B)$, $\tan(A + B)$, $\tan(A - B)$ in terms of the trigonometric ratios of A and B. (*b*) Deduce these values. (*c*) Express them in words.

2. (*a*) Express in terms of the trigonometric ratios of A *each* of the following: $\sin 2A$, $\cos 2A$ (*three* different forms), $\tan 2A$, $\cot 2A$. (*b*) Derive these expressions.

3. (*a*) Show that $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$. (*b*) Show that $\cos 2A + \cos 2B = 2\cos(A + B)\cos(A - B)$. (*c*) State and derive an equivalent expression for the difference of two sines; (*d*) for the difference of two cosines.

4. Show how to find $\cos\frac{1}{2}A$ when $\cos A$ is known. Explain the ambiguity in the result. Determine the sign of the result when A is an angle in the third quadrant. Find the cosine of 112° 30′. [From $\cos 225°$.]

5. Prove $2\cos\dfrac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$ if A is between 270° and 360°.

6. Derive an expression for each of the following: $\sin 3A$ in terms of $\sin A$, $\cos 3A$ in terms of $\cos A$, $\tan 3A$ in terms of $\tan A$. [SUGGESTION: $3A = 2A + A$. See Art. 93.]

7. (*a*) If $\tan A = \frac{\sqrt{3}}{4-\sqrt{3}}$ and $\tan B = \frac{\sqrt{3}}{4+\sqrt{3}}$, prove that $\tan(A-B) = .375$.

(*b*) If $\sin A = \frac{1}{\sqrt{10}}$ and $\cos B = \frac{3}{5}$, find the value of $\tan(A+B)$. (*c*) Find $\tan(A+B)$, given that $\sin A = \frac{8}{17}$, $\sin B = \frac{5}{13}$.

8. (*a*) If $\tan\frac{A}{2} = \frac{a}{b}$, show that $\sin A = \frac{2ab}{a^2+b^2}$, $\sin 2A = \frac{4ab(a^2-b^2)}{(a^2+b^2)^2}$.

(*b*) If $\tan A = \frac{b}{a}$, prove that $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \frac{2\cos A}{\sqrt{\cos 2A}}$.

9. (*a*) Find $\sin 45°$, and thence deduce the ratios of $22° 30'$. (*b*) Prove that $\tan 67° 30' = 1 + \sqrt{2}$. (*c*) Deduce the ratios of $67° 30'$, (i.) from ratios of $45°$, $22° 30'$, (ii.) from ratios of $135°$.

10. (*a*) Given $\sin 30° = \frac{1}{2}$ and $\cos 45° = \frac{1}{2}\sqrt{2}$; find $\sin 15°$, $\cos 75°$. (*b*) Given $\sin 30° = \frac{1}{2}$; find the numerical values of the other ratios of $30°$; thence derive the ratios of $15°$, thence derive the ratios of $75°$, $105°$, $165°$, $195°$. (*c*) Prove the following: $\tan 15° + \tan 75° = 4$, $\cos 15° \cdot \cos 75° = .25$, $\sin 105° + \cos 105° = \cos 45°$, $\tan 15°(\tan 60° - \tan 30°) = \tan 60° + \tan 30° - 2$.

11. (*a*) Express $\sin 8A + \sin 2A$ as a product. (*b*) Express as a sum or difference: (i.) $2\cos A \cos B$, (ii.) $2\sin 50° \cos 20°$. (*c*) Prove without using tables that (i.) $\sin 70° - \sin 10° = \cos 40°$, (ii.) $\cos 20° + \cos 100° + \cos 140° = 0$. Verify by the tables.

12. Show that: (1) $\cot A \cot B \cos(A+B) = \cos A \cos B(\cot A \cot B - 1)$; (2) $\cos(A+B)\cos A + \sin(A+B)\sin A = \cos B$; (3) $\cos A - \sin A = \sqrt{2}\cos(A+45°)$; (4) $2\cos^8 x - 2\sin^8 x = \cos 2x(1+\cos^2 2x)$; (5) $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$; (6) $\cos^2 A - \cos A \cos(60° + A) + \sin^2(30° - A) = .75$; (7) $\tan\frac{1}{2}\theta = \sin\theta : 1 + \cos\theta$; (8) $\cos(135° + A) + \sin(135° - A) = 0$; (9) $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$.

13. Prove that: (1) $\frac{\sin x + \sin y}{\cos x - \cos y} = -\cot\frac{1}{2}(x-y)$; (2) $\tan\frac{A}{2} = \sqrt{\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A}}$; (3) $\tan(60° + A) - \tan(60° - A) = \frac{8\cot A}{\cot^2 A - 3}$; (4) $\frac{\cos(A+45°)}{\cos(A-45°)} = \sec 2A - \tan 2A$; (5) $\frac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B} = \tan(A-B)$; (6) $\sec 2A - \frac{1}{2}\tan 2A \sin 2A = \frac{\cot^2 A + \tan^2 A}{\cot^2 A - \tan^2 A}$.

14. (*a*) Find values of θ not greater than $180°$, which satisfy $\cot\theta = \tan\frac{\theta}{2}$. (*b*) Give all the positive angles less than $360°$, which satisfy the equation $\sin 2A = \sqrt{3}\cos 2A$.

15. Show that the value of $\sin(n+1)B\sin(n-1)B + \cos(n+1)B\cos(n-1)B$ is independent of n.

16. The cosines of two angles of a triangle ABC are $\frac{3}{5}$ and $\frac{12}{13}$, respectively; find all the trigonometric ratios of the third angle without using tables. Verify the results by means of the tables.

17. Two towers whose heights respectively are 180 and 80 ft., stand on a horizontal plane; from the foot of each tower the angle of elevation of the other is taken, and one angle is found to be double the other; prove that the horizontal distance between the towers is 240 ft., and show that the sine of the greater angle of elevation is .6.

CHAPTER VII.

1. In a triangle ABC, show that (1) $\sin(A+B)=\sin C$, (2) $\cos(A+B)=-\cos C$, (3) $\sin\frac{A+B}{2}=\cos\frac{C}{2}$, (4) $\cos\frac{A+B}{2}=\sin\frac{C}{2}$.

2. (*a*) State and prove the *Law of Sines* for the plane triangle. (*b*) State and prove the *Law of Cosines* for the plane triangle. (*c*) If the sines of the angles of a triangle are in the ratios of 13 : 14 : 15, prove that the cosines are in the ratios of 39 : 33 : 25.

3. (*a*) Prove that in ABC, $b+c:b-c=\tan\frac{1}{2}(B+C):\tan\frac{1}{2}(B-C)=\cot\frac{1}{2}A:\tan\frac{1}{2}(B-C)$. (*b*) Write and derive the expressions for the cosine of an angle of a triangle, and the cosine and the sine of half that angle, in terms of the sides of the triangle. (*c*) In the triangle ABC derive the formulas expressing $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$, in terms of a, b, c. (*d*) Prove that in any triangle ABC, $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

4. (*a*) Show how to solve a triangle when the three sides are given, (i.) without logarithms, (ii.) with logarithms. Derive all the formulas necessary. (*b*) Do the same when two sides and their included angle are given. (*c*) Do the same when two angles and a side are given.

5. (*a*) Explain carefully, and illustrate by figures, the case in which the solution of a triangle is ambiguous. (*b*) Write formulas for a complete solution and check, of a triangle, when two sides and an angle opposite to one of them are given. How many solutions are there? Discuss fully all cases that may arise. (*c*) Given the angle A, and the sides a and b of a triangle ABC, determine whether there will be one solution, two solutions, or no solution, in each of the following cases: (i.) $A<90°$, $a>b$, (ii.) $A<90°$, $a=b$, (iii.) $A<90°$, $a<b$, (iv.) $A>90°$, $a>b$, (v.) $A>90°$, $a=b$.

6. Show by the trigonometric formulas that the angles of a triangle can be found when the ratios of the three sides are given. Give the geometrical explanation.

7. Show by the trigonometric formulas that the other two angles of a triangle can be found when the third angle, and the ratio of the sides containing it, are known. Give the geometrical explanation.

8. Assuming the law of sines for a plane triangle, prove that $a+b:c=\cos\frac{1}{2}(A-B):\sin\frac{1}{2}C$, and $a-b:c=\sin\frac{1}{2}(A-B):\cos\frac{1}{2}C$.

9. (*a*) If $A:B:C=2:3:4$, prove $2\cos\frac{A}{2}=\frac{a+c}{b}$. (*b*) If $2a=b+c$, prove $\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{3}$. (*c*) If a, b, c, the sides of a triangle, be in arithmetical progression, prove that $2\sin\frac{A}{2}\sin\frac{C}{2}=\sin\frac{B}{2}$. [SUGGESTION: Put $2b=a+c$.]

10. [In each of the examples in Ex. 10, A, B, C, denote the angles, a, b, c, the sides of the triangle.] Solve the following triangles (1) by making an estimate, (2) by the method of construction, (3) by computation, without using the logarithms of the trigonometric ratios (four-place tables), (4) by computation, using logarithms (five-place tables), (5) by dividing some of the oblique triangles into right-angled triangles. Check the results by computation. When a solution is impossible, or ambiguous, explain why it is so. (1) $a=753$, $b=621$, $c=937$; (2) $a=9$, $b=17$, $c=14$; (3) $a=1236.5$, $b=1674.8$, $c=2532.7$; (4) $a=30$, $b=42$, $c=36$; (5) $a=621$, $b=237$, $c=325$; (6) $a=1237$, $b=1014$, $A=39°\,42'$; (7) $a=1114$, $b=1345$, $A=46°\,54'\,20''$; (8) $c=832$, $b=694$, $B=54°\,47'\,30''$; (9) $a=1020$, $b=240$, $B=70°\,25'$; (10) $c=794$, $b=832$, $B=65°\,30'\,20''$; (11) $c=230$, $a=950$, $C=63°\,47'$; (12) $a=237$, $c=452$, $C=37°\,49'$; (13) $a=420$, $c=337$, $C=42°\,46'$; (14) $a=452$, $b=624$, $C=37°\,23'$; (15) $a=1237.4$, $c=1941.6$, $B=23°\,41'\,20''$; (16) $b=237.41$, $c=556.82$, $A=85°\,45'\,35''$; (17) $A=37°\,41'$, $B=49°\,32'$, $c=385.9$; (18) $B=47°\,21'\,30''$, $C=81°\,49'\,45''$, $b=374.26$.

11. (*a*) If $b:c=11:15$, and $A=37°\,40'$, find B and C. (*b*) If one side of a triangle be five times the other, and their included angle be 64°, find the remaining angles.

12. (*a*) In ABC, if $a:b:c=8:7:5$, find the angles. (*b*) The sides of a triangle are proportional to the numbers 4, 5, 6. Find the least angle.

13. Given $a=2b$, $C=120°$, find A, B, and the ratio $c:a$.

14. (*a*) Two adjacent sides of a parallelogram are respectively equal to 12 and 20 in., and a diagonal is equal to 25 in. Find the angles of the parallelogram, the other diagonal, and the area. (*b*) The sides of a quadrilateral taken in order are 8, 10, 16, 18, and one diagonal is 18. Find its angles and area.

15. A ladder 52 ft. long is set 20 ft. in front of an inclined buttress, and reaches 46 ft. up its face. Find the inclination of the face of the buttress.

16. A privateer is lying 10 mi. W.S.W. of a harbour, when a merchantman leaves it, steering S.E. 8 mi. an hour. If the privateer overtakes the merchantman in 2 hr., find her course and rate of sailing.

17. A fort bore E. by N. from a beacon, and was distant from it 1500 yd. From a ship at anchor the beacon bore N.N.W. and the fort N.E. by N. How far was the ship from the beacon?

18. A and B are two points, 200 yd. apart, on the bank of a river, and C is a point on the opposite bank; the angles ABC, BAC are respectively 54° 30′ and 65° 30′. Find the breadth of the river.

19. (*a*) Two observers on the same side of a balloon, and in the same vertical plane with it, are a mile apart, and they find the angles of elevation to be 22° 18′ and 75° 30′, respectively. What is the height? (*b*) Two observers on opposite sides of a balloon, and in the same vertical plane with it, take its altitude simultaneously; one observer finds it to be 64° 15′, and the other, 48° 20′. Find the height of the balloon at the time of observation.

20. From a ship sailing along a coast a headland, C, was observed to bear N.E. by N. After the ship had sailed E. by N. 15 mi. the headland bore W.N.W. Find the distance of the headland at each observation.

21. From a certain station a fort, A, bore N., and a second fort, B, N.E. by E. Guns are fired simultaneously from the two forts, and are heard at the station in 1.5 sec. and 2 sec. respectively. Assuming that sound travels at the rate of 1142 ft. per second, find the distance of the two forts apart.

22. From a point A in the same plane as the base of a tower, the tower bears N. 62° W., and the angle of elevation of the top of the tower is 53° 37′; from B, 165 ft. due north of A, the tower bears west. What is the height of the tower?

23. From a ship steering W. by S. a beacon bore N.N.W., and after the ship had sailed 12 mi. farther, the bearing of the beacon was N.E. by E. At what distance had the ship passed the beacon?

24. From the intersection of two straight paths which are inclined to each other at an angle of 37°, two pedestrians, A and B, start at the same instant to walk along the paths, A at the rate of 5 mi. an hour, and B at a uniform rate also; after 3 hr. they are 9½ mi. apart. Show that there are two rates at which B may walk to fulfil this condition, and find both of those rates.

25. Two straight railroads are inclined to each other at an angle of 22° 15′. At the same instant two engines, A and B, start from a station at the point of intersection, A going on one road at the rate of 20 mi. an hour, and B going uniformly on the other. After 3 hr. A and B are 25 mi. apart. Show that there are two rates at which B may go to fulfil this condition, and find those rates.

26. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°; a line was measured straight up the incline from the foot of the tower of 100 ft. in length, and at the upper extremity of this line the tower subtended an angle of 54°. Find the height of the tower.

27. The altitude of a certain rock is observed to be 47°, and after walking toward it 1000 ft. up a slope inclined at 32° to the horizon, the observer finds that this altitude is 77°. Find the vertical height of the rock above the first point of observation.

28. If, from a point at which the elevation of the observatory on Ben Nevis is 60°, a man walks 800 ft. on a level plane toward the mountain, and then 800 ft. further up a slope of 30° to a point at which the elevation of the observatory is 75°, show that the height of Ben Nevis is approximately 4478 ft., the man's path being always supposed to lie in a vertical plane passing through the observatory.

29. A man walks 40 ft. in going straight down the slope of the embankment of a railway which runs due east and west, and then walks 20 ft. along the foot of the embankment; he finds that he is exactly N.E. of the point from which he started at the top of the bank. Show that the inclination of the bank to the horizon is 60°.

30. A man in a ship at sea sailing north observes two rocks, A and B, to bear 25° east of his course; he then sails in a direction northwest for 4 mi., and observes A to bear east and B northeast of his new position. Find the distance from A to B.

31. To determine the distance of two forts, C, D, at the mouth of a harbour, a boat is placed at A, with its bow toward a distant object E, and the angles CAD, DAE are observed and found to be 22° 17′ and 48° 1′ respectively. The boat is then rowed to B, a distance of 1000 yd., directly toward E, and the angles CBD, DBE are observed to be 53° 15′ and 75° 43′ respectively. Find the distance CD.

32. A fort stands on a horizontal plane; the angle of depression measured from the top of the fort to a point P on the plane is $A°$, and to a point B, a feet beyond P, is $B°$. Derive the formulas for computing h, the height of the fort, and d, the distance from P to the bottom of the fort.

33. A person standing on a level plain at the base of a hill wishes to find the height of a tower which is in full sight on the top of the hill. Describe in detail the necessary measurements and computations.

34. A surveyor wishes to find the distance and the height of a tower which is on the same level with him, but on the opposite side of an impassable chasm; illustrate the problem by a lettered figure, and describe in detail the necessary measurements and computations.

35. What measurements must be made by an observer on the shore to find the distance between two buoys? Give formulas necessary for solving.

36. Explain what measurements have to be made at two stations, A and B, in order to find the distance, CD, between two inaccessible objects (A, B, C, D being in one plane); and state clearly the steps of the calculation by which the distance is to be found therefrom.

37. (a) From the law of sines deduce that $b \cos C + c \cos B = a$ (b) Prove this geometrically. (c) Show that $a \cos B - b \cos A = \frac{a^2 - b^2}{c}$.

38. In any triangle ABC, prove: (a) $\tan B = \dfrac{b \sin C}{a - b \cos C}$;

(b) $\dfrac{a^2 + b^2 - ab \cos C}{a \sin A + b \sin B + c \sin C} = \dfrac{a}{2 \sin A}$; ($c$) $\dfrac{1 + \cos (A - B) \cos C}{1 + \cos (A - C) \cos B} = \dfrac{a^2 + b^2}{a^2 + c^2}$.

CHAPTER VIII.

[In what follows, S denotes the area of a triangle, s its semi-perimeter, R, r, r_a, r_b, r_c, the radii of its circumscribing, inscribed, and escribed circles, respectively.]

1. Prove that any side of a triangle is equal to the second side into the cosine of the angle opposite the third sine plus the third side into the cosine of the angle opposite the second side.

2. Derive expressions, in terms of the sides of a given triangle, for the radii of its circumscribing circle, and of the four circles which touch the sides.

3. Derive expressions for the radii in Ex. 2, in terms of the sides and the area of the given triangle.

4. Prove $R = \dfrac{abc}{4S}$, $r = \dfrac{S}{s}$, $r_a = \dfrac{S}{s - a}$. Write similar expressions for r_b, r_c.

5. Prove: (a) $r_a + r_b + r_c - r = 4R$; (b) $\sqrt{r \cdot r_a \cdot r_b \cdot r_c} = S$.

6. Prove: (a) $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$; ($b$) $Rr = \dfrac{abc}{4(a + b + c)}$;

(c) $r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$.

7. Prove $r_a \cot \dfrac{A}{2} = r_b \cot \dfrac{B}{2} = r_c \cot \dfrac{C}{2} = r \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$

$= 4R \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

8. Prove $R = \dfrac{a}{2 \sin A}$, $r = (s - a) \tan \dfrac{A}{2}$, $r_a = s \tan \dfrac{A}{2}$. Write two other similar formulas for R and r. Write similar formulas for $r_b r_c$.

9. (a) Prove $r = \dfrac{a \sin \dfrac{B}{2} \sin \dfrac{C}{2}}{\cos \dfrac{A}{2}}$. Write two similar formulas involving b, c.

(b) Prove $r_a = \dfrac{a \cos \dfrac{B}{2} \cos \dfrac{C}{2}}{\cos \dfrac{A}{2}}$. Write similar formulas for r_b, r_c.

10. Show that the length of the tangents to the inscribed circle from the angle A is $s - a$, from the angle B is $s - b$, from the angle C is $s - c$.

11. Write and derive the formula for the area of a triangle: (a) in terms of the three sides; (b) in terms of two sides and their included angle; (c) in terms of one side and the two angles adjacent to it.

12. (*a*) Prove that $S = s(s - c)$ when $C = 90°$; (*b*) if x, y, be the lengths of the two diagonals of a parallelogram, and θ the angle between them, show that area $= \frac{1}{2}xy \sin\theta$.

13. Find the areas of some of the triangles in Ex. 10, Chap. VII Find the radii of their circumscribing, inscribed, and escribed circles.

14. (*a*) An isosceles triangle whose vertical angle is 78° contains 400 square yards; find the lengths of the sides. (*b*) Find two triangles each of which has sides 63 and 55 ft. long, and an area of 874 sq. ft. (*c*) The angles at the base of a triangle are 22° 30′ and 112° 30′ respectively; show that the area of the triangle is equal to the square of half the base.

15. (*a*) Show that the area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed and circumscribing polygon of half the number of sides. (*b*) The sides of a triangle are as 2 : 3 : 4; show that the radii of the escribed circles are as $\frac{1}{2} : \frac{1}{3} : 1$.

16. Two roads form an angle of 27° 10′ 25″. At what distance from their intersection must a fence at right angles to one of them be placed so as to enclose an acre of land?

17. If the altitude of an isosceles triangle is equal to its base, the radius of the circumscribing circle is $\frac{5}{8}$ of the base.

18. An equilateral triangle and a regular hexagon have the same perimeter. Show that the areas of their inscribed circles are as 4 : 9.

19. If the sides of a triangle are 51, 68, and 85 ft., show that the shortest side is divided by the point of contact of the inscribed circle into two segments, one of which is double the other.

CHAPTER IX.

1. Explain how angles are measured (1) by sexagesimal measure, (2) by radian measure. Show how to connect the radian measure of an angle with its measure in degrees. Find the number of degrees in the angle called the radian. How many degrees are there in an arc whose length is equal to the diameter? Show that the radian measure of an angle is the ratio of the lengths of two lines. What advantage is there in using radian measure?

2. (*a*) Give the number of degrees in each of the following angles: $\frac{1}{2}\pi$, $\frac{5}{4}\pi$, 2π, $\frac{5}{16}\pi$, $n\pi$, $\frac{\pi}{3}$, $-\frac{2}{5}\pi$, $\frac{3}{8}\pi$, $\frac{2}{5}\pi$, $\frac{5}{2}\pi$, $\frac{7}{2}\pi$, $\frac{7}{6}\pi$, $\frac{5}{6}\pi$, $-\frac{\pi}{7}$, $-\frac{4}{5}\pi$, $-1\frac{1}{3}\pi$, $(\frac{5}{6})^{(r)}$, $(2\frac{1}{3})^{(r)}$, $(-\frac{4}{5})^{(r)}$. (*b*) Give the supplements and complements of those angles in radian measure and in degree measure. (*c*) Give the radian measures of 30°, 80°, 49°, 41° 30′ 15″, 120°, − 210°, − 175°. Give the radian measures of their supplements and complements.

3. (*a*) A central angle 1.25^r is subtended by a circular arc of 16 ft.; find the radius. (*b*) Find the number of radians and degrees in the central angle subtended by an arc 9 in. long, in a circle whose radius is 10 ft.

(*c*) Find the radius of a circle in which an arc 15 in. long subtends at the centre an angle containing 71° 36′ 3″.6. (*d*) If the radius be 8 in., find the central angle, in degrees and in radians, that is subtended by an arc 15 in. long. (*e*) An angle of 3^r is subtended by an arc of 5 in.; find the length of the radius; find also the number of radians, and of degrees, in an arc of 1.5 in. (*f*) Find the number of radians and seconds in the angle subtended at the centre of a circle whose radius is 2 mi., by an arc 11 in. long. (*g*) Find the length of the arc which subtends a central angle of (1) 2 radians, the radius being 10 in.; (2) 1.5 radians, radius 2 ft.; (3) 4.3 radians, radius 21 yd.; (4) 1.25 radians, radius 8 in.

4. The value of the division on the outer rim of a graduated circle is 5′, and the distance between the two successive divisions is .1 of an inch. Find the radius of the circle.

5. Show that the distance in miles between two places on the equator, which differ in longitude by 3° 9′, assuming the earth's equatorial diameter to be 7925.6 mi., is 217.954 mi.

6. (*a*) The difference of two angles is 10°, the radian measure of their sum is 2. Find the radian measure of each angle. (*b*) One angle of a triangle is π degrees, another is π grades. Show that the radian measure of the third angle is $\pi - \frac{19\pi^2}{1800}$. (*c*) If the number of degrees in an angle be equal to the number of grades in the complement of the same angle, prove that the radian measure of the angle is $\frac{5\pi}{19}$. (*d*) The angles of a triangle are in the ratios 1 : 2 : 3. Express their magnitudes in each of the three systems of angular measurement. (*e*) One angle of a triangle is 45°, another is 1.5 radians. Find the third, both in degrees and in radians. (*f*) Express in degrees and in radian measure the vertical angle of an isosceles triangle which is half of each of the angles at the base.

7. Prove the following statements, in which a denotes the length of a side of a regular polygon; P, the length of its perimeter; n, the number of its sides; r, the radius of the inscribed circle; R, the radius of the circumscribing circle:

$$a = 2R\sin\frac{\pi}{n} = 2r\tan\frac{\pi}{n};\quad P = 2nR\sin\frac{\pi}{n} = 2nr\tan\frac{\pi}{n};\quad R + r = \frac{a}{2}\cot\frac{\pi}{2n};$$

$$\text{area of polygon} = \frac{n}{2}R^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{na^2}{4}\cot\frac{\pi}{n}.$$

CHAPTER X.

1. Define and illustrate the trigonometric functions. Show in tabular form the signs of these functions in each of the four quadrants.

2. (*a*) Construct a table showing the values, with proper signs, of the trigonometric functions of 0°, 30°, 45°, 60°, 120°, 180°, 225°, 270°, 315°, 360°.

(*b*) Compare the trigonometric functions of $90° - A$, $90° + A$, $180° + A$, $180° - A$, $-A$, with those of A.

3. Show, from both the ratio and the line definitions of the trigonometric functions, that (1) the sine and cosine are never greater than unity, (2) the cosecant and secant are never less than unity, (3) the tangent and cotangent may have any values whatever from negative infinity to positive infinity, (4) the trigonometric functions change signs in passing through zero or infinity, and through no other values.

4. Given $\tan A = -\frac{8}{15}$, find the values of the other trigonometric functions of A.

5. Find geometrically an expression for the cosine of the difference of two angles in terms of the trigonometric functions of those angles.

6. Prove that:

(*a*) $\sin^2 B + \sin^2 (A - B) + 2 \sin B \sin (A - B) \cos A = \sin^2 A$;

(*b*) $\dfrac{\cos nA - \cos (n + 2)A}{\sin (n + 2)A - \sin nA} = \tan (n + 1)A$.

7. Give the ratio definitions of the trigonometric functions, sine, cosine, tangent, and secant. These functions have also been defined as straight lines. Give these definitions, and show from them that tan 90°, sec 90° would each be infinite. *Show that the two systems are consistent.*

8. (*a*) Trace the changes, in magnitude and sign, in the values of the trigonometric functions as the angle increases from 0° to 360°. (*b*) Trace the changes of sign of $\sin \theta$ as θ increases through 360°, and show that its equivalent $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ has always the same sign as $\sin \theta$.

9. Trace the changes, as A increases from 0° to 180°, in the sign and value of (*a*) $\cos (\pi \sin A)$, (*b*) $\sin A + \cos A$, (*c*) $\sin A - \cos A$. Draw the graphs of these functions.

10. Show that the radian measure of an acute angle is intermediate in value between the sine and the tangent of the angle.

11. (*a*) Show that the limit of $\dfrac{\sin \theta}{\theta}$ when θ is indefinitely diminished is 1, *i.e.* $\sin \theta = \theta$, very nearly. (*b*) Show that the limit of $\dfrac{\tan \theta}{\theta}$ when θ is indefinitely diminished is 1, *i.e.* $\tan \theta = \theta$, very nearly.

12. Find the area of a regular polygon of n sides inscribed in a circle, and show, by increasing the number of sides of the polygon without limit, how the expression for the area of the circle may be obtained.

13. (*a*) Find the distance at which a building 50 ft. wide will subtend an angle of 3′. (*b*) A church spire 45 ft. high subtends an angle of 9′ at the eye. Find its distance approximately. (*c*) Find approximately the distance of a tower 51 ft. high which subtends at the eye an angle of $5\frac{5}{11}'$. (*d*) How large a mark on a target 1000 yd. off will subtend an angle of 1″ at the eye?

14. Show how the functions may be represented by lines connected with a circle.

15. Explain, with illustrations, how functions may be graphically represented by means of a curve. Draw the graphs of the trigonometric functions.

NOTE. — "If a function of a variable has its magnitude unaltered when the sign of the variable is changed, that function is called *an even function*, but if the function has the same numerical value as before, but with opposite sign, then that function is called *an odd function;* for instance, x^2 is an even function of x, x^3 is an odd function of x, but $x^2 + x^3$ is neither even nor odd, since its numerical value changes when the sign of x is changed."

16. Show that the cosine, secant, and versine of an angle are even functions, and the sine, tangent, cotangent, and cosecant are odd functions, and the coversine is neither even nor odd. (See Art. 78.)

CHAPTER XI.

N.B. The problems which are purely numerical are to be solved independently of tables. The results can be verified by means of the tables.

1. (*a*) Deduce a *general expression* for all angles which have the same *sine;* (*b*) for all which have the same *cosine;* (*c*) for all which have the same *tangent.* (*d*) What are the general expressions for all angles which have the same *secant*, *cosecant*, *cotangent*, respectively?

2. Define inverse trigonometric functions; give illustrations. Define $\tan^{-1} x$, $\cos^{-1} x$.

3. (*a*) Explain fully the equations $\sin(\sin^{-1}\frac{1}{3}) = \frac{1}{3}$, $\sin^{-1}(\sin\theta) = \theta$. (*b*) Construct $\sin^{-1}(\frac{2}{3})$, $\cos^{-1} 0$, $\tan^{-1}\infty$, $\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$. (*c*) Find $\tan(\cos^{-1}\frac{2}{3})$.

[Carefully state the limitations under which the following equations are true.]

4. Show that: (*a*) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$; (*b*) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$; (*c*) $\sin^{-1} x - \sin^{-1} y = \cos^{-1}\sqrt{1 - x^2 - y^2 + x^2y^2} + xy$.

5. (*a*) From $\sin 2A = 2\sin A\cos A$, show that $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$. (*b*) Show that for certain values of the angles, $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$; $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$; $2\cot^{-1} x = \operatorname{cosec}^{-1}\frac{1+x^2}{2x}$.

6. Show that for certain values of the angles: (*a*) $\cos^{-1} x = \sin^{-1}\sqrt{\frac{1-x}{2}} + \cos^{-1}\sqrt{\frac{1+x}{2}}$; (*b*) $\sin^{-1}\sqrt{\frac{x}{a+x}} = \tan^{-1}\sqrt{\frac{x}{a}} = \frac{1}{2}\cos^{-1}\frac{a-x}{a+x}$. (*c*) $\tan^{-1} m + \cot^{-1} m = \frac{\pi}{2}$, or $\frac{3\pi}{2}$.

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7. Prove that for certain values: (1) $\sec^{-1}3 = 2\cot^{-1}\sqrt{2}$; (2) $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3) = 15$; (3) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = 90°$; (4) $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}} = \frac{\pi}{6}$; (5) $\cos^{-1}\frac{44}{125} + 2\tan^{-1}\frac{1}{3} = \sin^{-1}\frac{4}{5}$; (6) $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = 120°$; (7) $\tan^{-1}\frac{1}{3} + \cot^{-1}\frac{1}{3} + \sin^{-1}\frac{7\sqrt{2}}{10} = 0$; (8) $\tan^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} + \tan^{-1}\sqrt{\frac{3}{2}} = \frac{3\pi}{4}$.

8. Prove that: (1) $\sin^{-1}\frac{2mn}{m^2+n^2} + \sin^{-1}\frac{m^2-n^2}{m^2+n^2} = \frac{\pi}{2}$; (2) $\tan^{-1}(\cot A) - \tan^{-1}(\tan A) = n\pi + \frac{\pi}{2} - 2A$; (3) $\tan^{-1}t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2}$; (4) $\tan^{-1}\frac{m-1}{m} + \tan^{-1}\frac{1}{2m-1} = n\pi + \frac{\pi}{4}$; (5) $\tan^{-1}\frac{2a-b}{b\sqrt{3}} + \tan^{-1}\frac{2b-a}{a\sqrt{3}} = \frac{\pi}{3}$; (6) $\tan^{-1}m + \tan^{-1}n = \cos^{-1}\frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}$.

9. Prove that: (1) $\tan^{-1}\frac{1}{1+a} + \tan^{-1}\frac{1}{1-a} + \tan^{-1}\frac{2}{a^2} = n\pi$; (2) $\tan^{-1}1 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$; (3) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{9} = 45°$; (4) $\cot^{-1}\frac{1}{3} = \cot^{-1}3 + \cot^{-1}\frac{3}{4}$; (5) $\cot^{-1}\frac{1-a}{1+a} - \cot^{-1}\frac{1-b}{1+b} = \sin^{-1}\frac{a-b}{\sqrt{1+a^2}\sqrt{1+b^2}}$; (6) $4(\cot^{-1}\frac{1}{3} + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$; (7) $a\cos\left(\sin^{-1}\frac{b}{a}\right) = \sqrt{a^2-b^2}$; (8) $\sin^{-1}\left(\frac{x-a+b}{2b}\right)^{\frac{1}{2}} = \frac{1}{2}\cos^{-1}\frac{a-x}{b}$; (9) $\sin\cot^{-1}a = \tan\cos^{-1}\sqrt{\frac{a^2+1}{a^2+2}}$; (10) $\{\tan(\sin^{-1}a) + \cot(\cos^{-1}a)\}^2 = 2a\tan(2\tan^{-1}a)$.

10. Find all the angles (*i.e.* find the general values of the angles) which satisfy the following equations: (1) $\sec^2 A = \frac{4}{3}$; (2) $2\tan^2\theta = \sec^2\theta$; (3) $\tan^2\theta - \sec\theta = 1$; (4) $\sqrt{3}\tan^2\theta + 1 = (1+\sqrt{3})\tan\theta$; (5) $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$; (6) $2\sin x + 2\operatorname{cosec} x = 5$; (7) $2\sin 2y = 3\tan y$; (8) $\cos B + \tan B = \sec B$; (9) $3\cos^2 x + 2\sqrt{3}\cos x = 5.25$; (10) $\tan z - 2\sin z = 0$; (11) $4\cot 2\theta = \cot^2\theta - \tan^2\theta$; (12) $\operatorname{cosec} C + \cot C = \sqrt{3}$; (13) $\cot A - \tan A = 2$; (14) $\operatorname{cosec} x = \cot x + \sqrt{3}$.

11. Solve $\cos 2x = \sin x$.

[SOLUTION: $\cos 2x = \cos\left(\frac{\pi}{2} - x\right)$; or, $\sin\left(\frac{\pi}{2} - 2x\right) = \sin x$. $\therefore \frac{\pi}{2} - x = 2n\pi \pm 2x$, or $\frac{\pi}{2} - 2x = n\pi + (-1)^n x$.]

12. Solve $\sin 5A = \sin 11A$.

[SOLUTION: $11A = n\pi + (-1)^n 5A$. $\therefore A = 0, \frac{\pi}{16}, \frac{\pi}{3}, \ldots$.]

13. Find the general solutions of: (1) $\sin\theta + \cos\theta = \sqrt{2}$; (2) $\sin 4\theta = \sin\theta$; (3) $2\cos 2\theta - 2\sin\theta - 1 = 0$; (4) $\tan^2\theta + 3\cot^2\theta = 4$; (5) $\sin^4 x -$

$\cos^4 x = 1$; (6) $\sin^2 2x - \sin^2 x = .25$; (7) $\tan B + \cot B = 2$, $\cos y + \cos 2y + \cos 3y = 0$ (SUGGESTION: $\cos y + \cos 3y = 2\cos y \cos 2y$); (8) $\sin \frac{1}{2}x = \operatorname{cosec} x - \cot x$; (9) $\cos x + \cos 7x = \cos 4x$; (10) $\cos A + \cos \frac{1}{2}A = \cos \frac{1}{4}A$; (11) $\operatorname{cosec} z = 2 \sin z$; (12) $2 \tan^{-1} \cos A = \tan^{-1} 2 \operatorname{cosec} A$; (13) $\tan(A - 15°) = \frac{1}{3}\tan(A + 15°)$; (14) $\tan(45° + B) = 1 + \tan B$.

14. Find x in the following equations: (1) $\cos^{-1} x + \cos^{-1}(1 - x) = \cos^{-1}(-x)$; (2) $\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} \frac{3\sqrt{3}}{5}$; (3) $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1} \frac{8}{31}$; (4) $\tan^{-1} \frac{x+1}{x-1} + \tan^{-1} \frac{x-1}{x} = \tan^{-1}(-7)$; (5) $\tan^{-1} \frac{x-1}{x+1} + \tan^{-1} \frac{2x-1}{2x+1} = \tan^{-1} \frac{23}{36}$.

15. A flagstaff a feet high is on a tower $3a$ feet high; prove that, if the observer's eye is on a level with the top of the staff, and the staff and tower subtend equal angles, the observer is at a distance $a\sqrt{2}$ from the top of the flagstaff.

16. In any triangle ABC, if $\tan \frac{A}{2} = \frac{5}{6}$, and $\tan \frac{B}{2} = \frac{20}{37}$, find $\tan C$ without tables. Verify the result by means of the tables. Show that in such a triangle, $a + c = 2b$.

CHAPTER XII.

1. Explain the advantages of measuring angles by the *sexagesimal*, *centesimal*, and *radian* methods, respectively.

2. Show that, if x be the radian measure of a positive angle less than $\frac{\pi}{2}$, then (a) $\cos x$ is less than 1 but greater than $1 - \frac{1}{2}x^2$; (b) $\sin x$ is less than x but greater than $x - \frac{1}{4}x^3$. By means of (b) show how the sine of $10''$ may be calculated approximately.

3. (a) Express in terms of functions of A, *each* of the following: $\sin 2A$, $\cos 2A$ (*three* different forms), $\tan 2A$. (b) Find $\cos 3A$ in terms of $\cos A$. (c) Find $\sin 3A$ in terms of $\sin A$. (d) Find $\tan 3A$ in terms of $\tan A$, and from the formula determine the numerical value of $\tan A$ if $3A = 90°$. (e) Investigate a formula for expressing the cosine of half an angle in terms of the sine of the whole angle; and if the angle lies between 270° and 360°, show which signs of the roots must be taken.

4. Show that $\sin(A + B - C) + \sin(A + C - B) + \sin(B + C - A) - \sin(A + B + C) = 4 \sin A \sin B \sin C$.

5. If $A + B + C = 180°$ (*i.e.* if A, B, C be the three angles of a triangle), show that: (a) $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$; ($b$) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$; ($c$) $\frac{\cos A + \cos B + \cos C - 1}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.

6. If $\sin A = \frac{3}{5}$, $\sin B = \frac{12}{13}$, and $\sin C = \frac{7}{25}$, where A, B, C are positive angles and less than 90°, find $\sin(A + B + C)$.

7. Assuming the equation $\cos 3x = 4\cos^3 x - 3\cos x$, find $\sin 18°$.

[SOLUTION: $54° + 36° = 90°$. $\therefore \cos 54° = \sin 36°$; *i.e.* $\cos 3 \cdot 18° = \sin 2 \cdot 18°$. Hence, $4\cos^3 18° - 3\cos 18° = 2\sin 18°\cos 18°$. $\therefore 4\cos^2 18° - 3 = 2\sin 18°$. On putting $1 - \sin^2 18°$ for $\cos^2 18°$, and solving for $\sin 18°$, there is obtained the result, $\sin 18° = \frac{\sqrt{5} - 1}{4}$.]

8. Assuming the result in Ex. 7, find the other trigonometric functions of 18° and the functions of 72°.

9. Assuming the result in Ex. 7, show that $\cos 36° = \frac{\sqrt{5} + 1}{4} = \sin 54°$. Hence, deduce the other trigonometric functions of 36° and 54°. Also, deduce the trigonometric functions of 9° and 81°. (The results in Exs. 8, 9, can be verified by means of the tables.)

10. Prove the formulas:

$$\sin(36° + A) - \sin(36° - A) - \sin(72° + A) + \sin(72° - A) = \sin A,$$
$$\cos(36° + A) + \cos(36° - A) - \cos(72° + A) - \cos(72° - A) = \cos A,$$

and explain their use. (See Art. 97.)

11. (*a*) Show that $\sin^2 30° = \sin 18° \sin 54°$. (*b*) Solve $x^3 \cot 108° = 128 \sin 72° \cos 18°$, without tables. (*c*) Find the trigonometric functions of 48°.

[HINT: $48° = 30° + 18°$.]

12. Two parallel chords of a circle lying on the same side of the centre of a circle subtend angles of 72° and 144° at the centre. Show that the distance between the chords is equal to half the radius of the circle, (*a*) using tables, (*b*) not using tables.

13. (*a*) Solve: (i.) $\cos\theta = 0$; (ii.) $\sin x + \cos x = 1$;
(iii.) $\tan y + \tan 4y + \tan 7y = \tan y \tan 4y \tan 7y$.

(*b*) If $\tan\theta \tan 3\theta = -.4$, find $\tan\theta$, $\tan 3\theta$.

14. (*a*) If in triangle ABC, $A = 3B$, show that $\sin B = \frac{1}{2}\sqrt{\frac{3b - a}{b}}$. (*b*) Given $\cos A = .28$, find $\tan\frac{A}{2}$. Explain the reason of the ambiguity that presents itself in the result. (*c*) If $\sin A = \frac{2ab}{a^2 + b^2}$, find $\tan\frac{A}{2}$. (*d*) Given $\tan\frac{1}{2}x = 2 - \sqrt{3}$, find $\sin x$.

15. Prove the following:

(i.) $\tan A - \tan\frac{1}{2}A = \tan\frac{1}{2}A \sec A$.
(ii.) $1 + \tan^6 A = \sec^4 A(\sec^2 A - 3\sin^2 A)$.
(iii.) $\sin A + \sin 3A + \sin 5A + \sin 7A = 16 \sin A \cos^2 A \cos^2 2A$.
(iv.) $\cos 6A = 16(\cos^6 A - \sin^6 A) - 15\cos 2A$.
(v.) $\sin 3x + \sin 5x = 8 \sin x \cos^2 x \cos 2x$.

(vi.) $4\cos^3 A \sin 3A + 4\sin^3 A \cos 3A = 3\sin 4A$.

(vii.) $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$.

(viii.) $\sin^3 A + \sin^3 (120° + A) + \sin^3 (240° + A) = -\frac{3}{4}\sin 3A$.

(ix.) $1 + \cos 2(A - B)\cos 2B = \cos^2 A + \cos^2 (A - 2B)$.

(x.) $2\operatorname{cosec} 4A + 2\cot 4A = \cot A - \tan A$.

16. Show that

$$\cos (36° + A)\cos (36° - A) + \cos (54° + A)\cos (54° - A) = \cos 2A;$$
$$\sin 3A = 4\sin A \sin (60° + A)\sin (60° - A).$$

17. Show that

$$2\cos\frac{A}{2} = \pm\sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}, \quad 2\sin\frac{A}{2} = \pm\sqrt{1 + \sin A} \mp \sqrt{1 - \sin A}.$$

[SUGGESTION: $\cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1$; $2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A$.]

18. Prove that the following equations are true for certain values of the angles:

(i.) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$.

(ii.) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$.

(iii.) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x + y + z - xyz}{1 - xy - yz - zx}$.

(iv.) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}\dfrac{1 - x - y - xy}{1 + x + y - xy} = \dfrac{\pi}{4}$.

(v.) Given $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, $\tan\gamma = \frac{1}{8}$, find $\tan(\alpha + \beta + \gamma)$.

(vi.) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \frac{1}{2}\cos^{-1}\frac{3}{5}$.

(vii.) $\tan^{-1}\dfrac{4}{3} = \dfrac{1}{2}\tan^{-1}\left(\dfrac{-24}{7}\right) = \dfrac{1}{3}\tan^{-1}\left(\dfrac{-44}{117}\right)$.

(viii.) $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \dfrac{\pi}{2}$.

(ix.) $\sin^{-1}\frac{24}{25} = 3\sin^{-1}\frac{1}{5}$.

(x.) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4} = \sin^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1}3$.

(xi.) $\sin^{-1}\dfrac{3}{\sqrt{73}} + \cos^{-1}\dfrac{11}{\sqrt{146}} + \sin^{-1}\dfrac{1}{2} = \dfrac{5\pi}{12}$.

(xii.) $\tan^{-1}\frac{2}{3} = \frac{1}{2}\tan^{-1}\frac{12}{5}$.

19. The hypotenuse and shortest side of a right-angled triangle are 5 ft. and 3 ft., respectively. Find the length of the perpendicular from the right angle upon the hypotenuse, and show that it is inclined at $\sin^{-1}\frac{7}{25}$ to the straight line drawn from the right angle to the middle point of the hypotenuse.

20. If a triangle ABC is to be solved from given parts A, a, b, show that the solution is sometimes ambiguous; and that in such a case the difference of the two values of C is $2\cos^{-1}\dfrac{b\sin A}{a}$.

21. The tangent of an angle is 2.4. Find the cosecant of the angle, the cosecant of half the angle, and the cosecant of the supplement of double the angle.

22. The angle of elevation of a tower at a distance of 20 yd. from its foot is three times as great as the angle of elevation 100 yd. from the same point; show that the height of the tower is $300 : \sqrt{7}$ ft.

23. DE is a tower on a horizontal plane. $ABCD$ is a straight line in the plane. The tower subtends an angle θ at A, 2θ at B, and 3θ at C. If $AB = 50$ ft., and $BC = 20$ ft., find the height of the tower and the distance CD.

24. A ship sailing at a uniform rate was observed to bear N. 30° 57′ 30″ E. After 20 minutes she bore N. 35° 32′ 15″ E., and after 10 minutes more, N. 37° 52′ 30″ E. Find the direction in which she was sailing.

[*Ans.* S. 44° 38′ E.]

25. A spectator observes the explosion of a meteor, due south of him, at an elevation of 28° 45′. To another spectator, 11 mi. S.S.W. of the former, it appears at the same instant to have an altitude of 42° 15′ 30″. Show that there are two possible heights above the earth's surface at which it may have exploded, and find these heights. [*Ans.* 4.33 mi. or 13.21 mi.]

ANSWERS TO THE EXAMPLES

IN

MURRAY'S PLANE TRIGONOMETRY

CHAPTER I.

Art. 2. **1.** $\log_3 27 = 3$, $\log_4 256 = 4$, $\log_{11} 121 = 2$, $\log_9 729 = 3$, $\log_7 343 = 3$, $\log_m p = b$. **2.** $2^3 = 8$, $5^4 = 625$, $10^3 = 1000$, $2^6 = 64$, $n^a = P$. **3.** 0, $\frac{1}{2}$, 2, 3, 4, 5, 6, 7, 8. **4.** 0, 1, 2, 3, 4, 5. **5.** 0, 1, 2, 3, 4, 5, 6, -1, -2, -3, -4, -5, -6. **6.** 1, 4, 16, 64, 256, 1024. **7.** 0, 1; 1, 2; 2, 3; 3, 4; 3, 4; 0, -1; -1, -2; -2, -3.

Art. 6. **6.** (*a*) 1.4007, (*b*) .09441, (*c*) .06906, (*d*) .09856. **7.** (*a*) 9.442, (*b*) 7.277, (*c*) 88.344. **8.** (*a*) 7.937, (*b*) 25.1, (*c*) 2.51, (*d*) .7937, (*e*) .251, (*f*) .07937. **9.** (*a*) 68.945, (*b*) 8.308, (*c*)15.674. **10.** 9.214, 2.691, .07051. **11.** .12927. **12.** .6443. **13.** (*a*) 3.236, (*b*) 2.64, (*c*) 1.56, (*d*)2.1038, (*e*) 2.88366, (*f*) 3.13481.

CHAPTER II.

Art. 8. **1.** 8, 24, 42, 54, 720, 36 *a*, 12 *b*, *c*. **2.** 6, 3.75, $2\frac{2}{3}$, $\frac{5}{6}$, $\frac{5}{18}$, $\frac{5}{36}$, a, $\frac{b}{3}$, $\frac{c}{36}$. **3.** 7920, 51, 9, 2, 8.5, 2.5, $\frac{3}{4}$, $\frac{1}{8}$, 3 *a*, *b*, $\frac{c}{12}$. **4.** 30, 6, $2\frac{2}{3}$, 12 *a*, 4 *b*, $\frac{c}{3}$. **5.** 440, $\frac{1}{54}$. **6.** 3 : 1. **7.** 6 in., $\frac{1}{2}$ in., $\frac{1}{8}$ in., 10 ft. **8.** 8; ratios are $\frac{4}{5}$, $\frac{3}{5}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{5}{3}$, $\frac{5}{4}$, respectively in each triangle. The angles in each triangle are respectively equal to the angles in the other triangles. **9.** 37; $\frac{12}{37}$, $\frac{35}{37}$, $\frac{12}{35}$, $\frac{35}{37}$, $\frac{35}{12}$, $\frac{37}{35}$, $\frac{37}{12}$; the same. The angles in the first triangle are respectively equal to the angles in the second triangle. **10.** $\frac{21}{29}$, $\frac{20}{29}$, $\frac{21}{20}$, $\frac{20}{21}$, $\frac{29}{20}$, $\frac{29}{21}$; the same. Angles are respectively equal.

Art. 9. **1.** .002. **2.** A and B, .006; C and A, $1\frac{32}{75}$. **3.** $\frac{11}{35}$ ft. **4.** $26\frac{107}{304}$ sq. ft. **5.** .17. **6.** 213.605 ft. **7.** 17.32 yd. **8.** 37.08 ft. **9.** 28.284 ft. **10.** 14.142. **11.** (*a*) .8660, .5, 1.7321, .5774, 2, .1155. (*b*) As in (*a*). Angles are respectively equal. **12.** (*a*) .8321, .5547, 1.5, .6667, 1.803, 1.202. (*b*) As in (*a*). Angles equal. **13.** (*a*) .4286, .904, .474, 2.108, 1.11, 2.3333. (*b*) As in (*a*). Angles equal.

Art. 10. **1.** .025, .05. .1, 1, .2, .5, 1.5, .75, 3, 4, 5.5 in. respectively. Scale 1 : 120. **2.** .4, .8, 1.4, 2.2, .733, .911 in. respectively. Scale 1 : 180.

3. 18.75, 37.5, 75, 262.5, 375, 720, 795 ft. respectively. Scale 1 : 1800. 4. 30, 70, 2.5, 5, 35 mi. respectively. 7, 1.8, 2.5 in. respectively. Scale 1 : 633600. 5. 1:1200; 1:253440; 1:1920; 1:8640; 1:63360; 1:633600; 1:6336000. 7. 2.446 mi.

Art. 11. 1. 36° 52′, 53° 8′; 18° 55′.6, 71° 4′.4; 46° 23′.5, 43° 36′.5; 30°, 60°; 33° 41′.4, 56° 18′.6; 25° 22′.7, 64° 37′.3. (In the following answers h, p, b, denote hypotenuse, perpendicular, and base, respectively.) 2. 35°, $h = 34.86$, $p = 28.56$, $\frac{b}{h} = .574$, $\frac{h}{b} = 1.743$, $\frac{p}{h} = .819$, $\frac{h}{p} = 1.22$, $\frac{p}{b} = 1.428$, $\frac{b}{p} = .7$. 3. 65°, $b = 27.19$, $p = 12.68$, $\frac{b}{h} = .906$, $\frac{h}{p} = 2.37$, $\frac{p}{h} = .423$, $\frac{h}{b} = 1.1$, $\frac{p}{b} = .466$, $\frac{b}{p} = 2.14$. 4. 56° 19′, 33° 41′ (nearly), $h = 54.08$, $\frac{b}{p} = .67$, $\frac{p}{b} = 1.5$, $\frac{h}{b} = 1.8$, $\frac{b}{h} = .555$, $\frac{h}{p} = 1.2$, $\frac{p}{h} = .833$. 5. 41° 25′, 48° 35′ (nearly), $p = 39.7$, $\frac{b}{h} = .75$, $\frac{h}{b} = 1.33$, $\frac{p}{h} = .66$, $\frac{h}{p} = 1.51$, $\frac{p}{b} = .88$, $\frac{b}{p} = 1.13$. 6. 50°, $p = 59.6$, $h = 77.8$, $\frac{b}{h} = .643$, $\frac{h}{b} = 1.56$, $\frac{p}{h} = .766$, $\frac{h}{p} = 1.31$, $\frac{p}{b} = 1.19$, $\frac{b}{p} = .839$.

Art. 12. 1. See Art. 16. 3. See answers to Ex. 8–10, Art. 8; Ex. 11–13, Art. 9; Ex. 2–6, Art. 11. 4. Sin $P = .7$, cos $P = .71414$, tan $P = .98$, cot $P = 1.0202$, sec $P = 1.403$, cosec $P = 1.4286$; sin $P = \cos R$, cos $P = \sin R$, tan $P = \cot R$, cot $P = \tan R$, sec $P = \operatorname{cosec} R$, cosec $P = \sec R$. 5. (1) 1, (2) 1, (3) 1, (4) 1, (5) 1, (6) 1, (7) 0, (8) 0. 6. Same results as in Ex. 5.

Art. 13. 3. .3035, .6873, .9732, .9932, .9502, .7915, .4754, .2978, .7722, 5.4623, 1.0913, .3167, 9.7828, 9.9740, 9.9361, 9.5034, 0.7322, 9.9492. 4. 8° 49′, 38° 26′.7, 55° 33′, 57° 28′.5, 30° 43′, 52° 3′, 32° 47′.5, 53° 0′.7, 23°1′, 31° 22′.5, 59° 16′, 68° 8′.4, 38° 18′.4, 61° 4′.3, 22° 20′.5, 56° 0′.7, 33° 31′.5, 67° 3′.6.

Art. 14. (In the following examples the ratios are given in the order on page 21.) 5. .8575, .5145, 1.6667, .6, 1.9437, 1.1662, 59° 2′.2. 6. .4, .9165, .4364, 2.2913, 1.0911, 2.5, 23° 35′; .3162, .9487, .3333, 3, 1.0541, 3.1623, 18° 25′.9. 7. 9035, .4286, 2.1082, .4743, 2.3333, 1.1068, 64° 37′.3. 11. (1) $a : b$, $\sqrt{b^2 - a^2} : b$, $a : \sqrt{b^2 - a^2}$, $\sqrt{b^2 - a^2} : a$, $b : \sqrt{b^2 - a^2}$, $b : a$; (2) $\sqrt{b^2 - a^2} : b$, $a : b$, $\sqrt{b^2 - a^2} : a$, $a : \sqrt{b^2 - a^2}$, $b : a$, $b : \sqrt{b^2 - a^2}$; (3) $a : \sqrt{a^2 + b^2}$, $b : \sqrt{a^2 + b^2}$, $a : b$, $b : a$, $\sqrt{a^2 + b^2} : b$, $\sqrt{a^2 + b^2} : a$; (4) $b : \sqrt{a^2 + b^2}$, $a : \sqrt{a^2 + b^2}$, $b : a$, $a : b$, $\sqrt{a^2 + b^2} : a$, $\sqrt{a^2 + b^2} : b$; (5) $\sqrt{a^2 - b^2} : a$, $b : a$, $\sqrt{a^2 - b^2} : b$, $b : \sqrt{a^2 - b^2}$, $a : b$, $a : \sqrt{a^2 - b^2}$; (6) $b : a$, $\sqrt{a^2 - b^2} : a$, $b : \sqrt{a^2 - b^2}$, $\sqrt{a^2 - b^2} : b$, $a : \sqrt{a^2 - b^2}$, $a : b$. 12. 41° 24′ 35″. 13. 19° 28′ 16″.

Art. 15. 1. 2.28025. 2. 2.3333. 3. 5.846. 4. 2.0404. 5. 2.25. 6. 4.619. 7. 5.9259. 8. .3248. 9. 2.75. 10. −.708.

Art. 16. 1. 80°, 77° 30′, 43°, 33° 32′ 25″. 2. 67°, 48°, 39°, 12°, 4°.

Art. 17. $A = \sin^{-1}\frac{2}{3}$, $x = \cos^{-1}\frac{3}{4}$, $C = \tan^{-1} 4$, $A = \sec^{-1} 9$, $A = \operatorname{cosec}^{-1}\frac{17}{10}$.

Art. 18. 9.

	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\operatorname{cosec} A$
$\sin A$	$\sin A$	$\sqrt{1-\cos^2 A}$	$\frac{\tan A}{\sqrt{1+\tan^2 A}}$	$\frac{1}{\sqrt{1+\cot^2 A}}$	$\frac{\sqrt{\sec^2 A-1}}{\sec A}$	$\frac{1}{\operatorname{cosec} A}$
$\cos A$	$\sqrt{1-\sin^2 A}$	$\cos A$	$\frac{1}{\sqrt{1+\tan^2 A}}$	$\frac{\cot A}{\sqrt{1+\cot^2 A}}$	$\frac{1}{\sec A}$	$\frac{\sqrt{\operatorname{cosec}^2 A-1}}{\operatorname{cosec} A}$
$\tan A$	$\frac{\sin A}{\sqrt{1-\sin^2 A}}$	$\frac{\sqrt{1-\cos^2 A}}{\cos A}$	$\tan A$	$\frac{1}{\cot A}$	$\sqrt{\sec^2 A-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2 A-1}}$
$\cot A$	$\frac{\sqrt{1-\sin^2 A}}{\sin A}$	$\frac{\cos A}{\sqrt{1-\cos^2 A}}$	$\frac{1}{\tan A}$	$\cot A$	$\frac{1}{\sqrt{\sec^2 A-1}}$	$\sqrt{\operatorname{cosec}^2 A-1}$
$\sec A$	$\frac{1}{\sqrt{1-\sin^2 A}}$	$\frac{1}{\cos A}$	$\sqrt{1+\tan^2 A}$	$\frac{\sqrt{1+\cot^2 A}}{\cot A}$	$\sec A$	$\frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A-1}}$
$\operatorname{cosec} A$	$\frac{1}{\sin A}$	$\frac{1}{\sqrt{1-\cos^2 A}}$	$\frac{\sqrt{1+\tan^2 A}}{\tan A}$	$\sqrt{1+\cot^2 A}$	$\frac{\sec A}{\sqrt{\sec^2 A-1}}$	$\operatorname{cosec} A$

15. 90°, 36° 52′ 12″. **16.** 45°. **17.** 45°, 71° 34′. **18.** 53° 7′ 48″. **19.** 30°, 48° 35′ 25″. **20.** 36° 52′ 12″, 16° 15′ 36′.

CHAPTER III.

Art. 27. 5. $A = 65° 14'$, $B = 24° 46'$, $b = 7.834$. **6.** 303.9, 39° 47′.6, 50° 12′.4. **7.** 58° 45′ 48″, $a = 1521.5$, $b = 2508.6$. **8.** $A = 21° 8'$, $b = 94.43$, $c = 101.24$. **9.** $A = 30° 12'.2$, $B = 59° 47'.8$, $c = 116.25$. **10.** $A = 1° 21'.9$, $B = 88° 38'.1$, $b = 45.95$. **11.** $B = 41° 43'$, $a = 241.85$, $b = 215.6$. **12.** $B = 38° 41'$, $a = 312.2$, $c = 400$. **13.** $B = 52° 39' 30''$, $a = 1040.9$, $b = 1364.3$. **14.** $A = 52° 37'.1$, $B = 37° 22'.9$, $c = 2912.1$. **15.** $B = 62° 14' 40''$, $a = 1968.7$, $c = 4227.4$. **16.** $A = 29° 24'.9$, $B = 60° 35'.1$, $b = 43.67$. **17.** $A = 27° 20'$, $b = 77.4$, $c = 87.1$. **18.** $b = 7.4833$, $A = 33° 44'.6$. **19.** $c = 8.75$, $A = 30° 57'.8$. **20.** $a = 9.57$, $b = 11.54$. **21.** $a = 3.84$, $b = 11.37$. **22.** $b = 8.9433$, $A = 41° 48'.6$. **23.** $a = 16.4$, $c = 22.2$. **24.** $c = 14.42$, $A = 33° 41'.4$.

CHAPTER IV.

Art. 28. 1. 24.95 ft., 12.71 ft. **2.** 58.78 in. **3.** (*a*) 9.239 in.; (*b*) 21.96 ft.; (*c*) 16.8 ft.; (*d*) 19.3 ft.

Art. 29. 4. 398.2 ft. **5.** 228.4, 258 ft. **6.** 63.88 ft. **7.** 276.95 ft. **8.** 467.2 ft. **9.** 3244 ft. **10.** 86.6, 50. **11.** 749.04 mi. **12.** 219.45 ft.

Art. 30. 1. 26.172, 52.345 mi.; second ship bears E. 19° 42′.1 N. from first. **2.** $LB = 14.197$ mi.

Art. 31. 2. 2392.18 sq. ft. 3. 2392.18 sq. ft. 4. 44361 sq. yd. 5. 22.5 sq. ft. 6. 435.7 sq. ft. 7. $\frac{1}{2} ca \sin B$, $\frac{1}{2} ab \sin C$.

Art. 32. 2. Base = 187.9 ft.; height = 350.63 ft.; area = 32943 sq. ft. 3. Base = 358.21 ft.; height = 161.25 ft.; area = 28881 sq. ft. 4. 68° 50′, 68° 50′, 42° 20′, 83.9 ft., 2727.6 sq. ft. 5. 36.056 ft., 56° 18′.6, 56° 18′.6, 67° 22′.8, 600 sq. ft. 6. 105.83 ft., 82° 49′.4, 48° 35′.3, 3174.9 sq. ft. 7. 44.9, 44.9, 40.76 ft., 815.2 sq. ft. 8. 79.4, 79.4, 96.7 ft., 3045.4 sq. ft.

Art. 33. 1. 14.54 ft., 16.13 ft., 48.45 sq. ft., 105.2 sq. ft. 2. 16.52 ft., 20.415 ft., 133.94 sq. ft., 318.4 sq. ft. 3. 28.97 ft., 31.4 ft. 4. 41.6, 12; 33.94, 16.97; 28.2, 19.4; 24, 20.8; 20.8, 21.6; 18.37, 22.2. 5. 83.14, 48, 34.87, 27.7, 23.1, 19.88.

Art. 34. a. 10.93, 106.5, 278.87.

Art. 34. b. 2. 10.954 mi. 3. 96 ft. 4. 14.454 mi. 5. 67.08 mi., dip = 57′ 39″. 6. 140.7 mi., dip = 2° 1′ 53″.

Art. 34. c. 3. 12.47 chains, S. 85° 16′ W.; 2.868 acres. 4. 2.852 acres. 5. 12 acres 3 roods 6.45 poles.

CHAPTER V.

Art. 36. 2. East 48 miles. 3. Zero. 4. South 102 miles.

Art. 37. 1. 137°, 65°, 321°, 210°, 120°, 340°, 280°, 183°, 57°; second, first, fourth, third, second, fourth, fourth, third, first; 137°, 497°, 857°, 1217°; 65°, 425°, 785°, 1145°; ···. 2. 120°, 23°, 113°, 250°, 330°, 30°, 277°; second, first, second, third, fourth, first, fourth; 120°, 480°, 840°, 1200°; 23°, 383°, 743°, 1103°; ···. 3. 220°, 10°, 220°, 220°.

Art. 38. 3. 320°, −60°, 130°, −250°, 15°, 7°, 78°, 385°, 414°, −110°, −150°, 200°, 257°. 4. 410°, 30°, 220°, −160°, 105°, 97°, 168°, 475°, 504°, −20°, −60°, 290°, 347°.

Art. 41. (The ratios are in the order given in Art. 12.) 2. Same as ratios of 60° (see Art 15). 3. Ratios of 120° and 480° are $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}, -\frac{1}{\sqrt{3}}, -2, \frac{2}{\sqrt{3}}$; ratios of 240° and 600° are $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}, \frac{1}{\sqrt{3}}, -2, -\frac{2}{\sqrt{3}}$; ratios of −60°, 300°, and 660° are $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}, -\frac{1}{\sqrt{3}}, 2, -\frac{2}{\sqrt{3}}$; ratios of −720° are 0, 1, 0, ∞, 1, ∞. 4. Of 150°: $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}, -\sqrt{3}, -\frac{2}{\sqrt{3}}, 2$; of 410°: .6428, .7660, .8391, 1.1918, 1.3, 1.5; of 210° and −1.50°: $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, -\frac{2}{\sqrt{3}}, -2$; of 330° and −390°: $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{2}{\sqrt{3}}, -2$. 5. Of 45° and 765°: $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, 1, \sqrt{2}, \sqrt{2}$; of 135° and −225°: $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1, -1, -\sqrt{2}, \sqrt{2}$; of 225° and 585°: $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1, 1, -\sqrt{2},$

$-\sqrt{2}$; of $-405°$ and $1035°$: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1, -1, \sqrt{2}, -\sqrt{2}$. **6.** Of $-754°$: $-.5592, .8290, -.6745, -1.4826, 1.2, -1.8$; of $487°$: $.7986, -.6018, -1.3270, -.7536, -1.6, 1.3$; of $-245°$: $.9063, -.4226, -2.1445, -.4663, -2.4, 1.1$. For **7, 8, 9, 10**, see Art. 45.

Art. 42. (Ratios are in the order in Art. 12. The upper signs are taken together, and the lower are taken together.) **3.** $\pm\frac{3}{5}, \mp\frac{4}{3}, -\frac{3}{4}, -\frac{4}{3}, \mp\frac{5}{4}, \pm\frac{5}{3}$. **4.** $\pm\frac{\sqrt{5}}{3}, \frac{2}{3}, \pm\frac{\sqrt{5}}{2}, \pm\frac{2}{\sqrt{5}}, \frac{3}{2}, \pm\frac{3}{\sqrt{5}}$. **5.** $\pm\frac{\sqrt{5}}{3}, -\frac{2}{3}, \mp\frac{\sqrt{5}}{2}, \mp\frac{2}{\sqrt{5}}, -\frac{3}{2}, \pm\frac{3}{\sqrt{5}}$. **6.** $\pm\frac{4}{5}, \pm\frac{3}{5}, \frac{4}{3}, \frac{3}{4}, \pm\frac{5}{3}, \pm\frac{5}{4}$. **7.** $-\frac{3}{5}, \pm\frac{4}{5}, \mp\frac{3}{4}, \mp\frac{4}{3}, \mp\frac{5}{4}, -\frac{5}{3}$. **8.** $\pm\frac{4}{5}, \frac{3}{5}, \pm\frac{4}{3}, \pm\frac{3}{4}, \frac{5}{3}, \pm\frac{5}{4}$. **9.** $\pm\frac{\sqrt{7}}{4}, -\frac{3}{4}, \mp\frac{\sqrt{7}}{3}, \mp\frac{3}{\sqrt{7}}, -\frac{4}{3}, \pm\frac{4}{\sqrt{7}}$. **10.** $-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, \mp\frac{1}{\sqrt{3}}, \mp\sqrt{3}, \pm\frac{2}{\sqrt{3}}, -2; \frac{4}{7}, \pm\frac{\sqrt{33}}{7}, \pm\frac{4}{\sqrt{33}}, \pm\frac{\sqrt{33}}{4}, \pm\frac{7}{\sqrt{33}}, \frac{7}{4}$.

Art. 44. **2.** $\pm\frac{\sqrt{5}}{3}, -\frac{2}{3}, \mp\frac{\sqrt{5}}{2}, \mp\frac{2}{\sqrt{5}}, -\frac{3}{2}, \pm\frac{3}{\sqrt{5}}$. **3.** $\pm\frac{5}{\sqrt{74}}, \pm\frac{7}{\sqrt{74}}, \frac{5}{7}, \frac{7}{5}, \pm\frac{\sqrt{74}}{7}, \pm\frac{\sqrt{74}}{5}$. **4.** $\pm\frac{\sqrt{15}}{4}, \frac{1}{4}, \pm\sqrt{15}, \pm\frac{1}{\sqrt{15}}, 4, \pm\frac{4}{\sqrt{15}}$. **5.** $-\frac{1}{5}, \pm\frac{\sqrt{24}}{5}, \mp\frac{1}{\sqrt{24}}, \mp\sqrt{24}, \pm\frac{5}{\sqrt{24}}, -5$. **6.** $-\frac{1}{7}, \pm\frac{4\sqrt{3}}{7}, \mp\frac{1}{4\sqrt{3}}, \mp 4\sqrt{3}, \pm\frac{7}{4\sqrt{3}}, -7$. **7.** $\pm\frac{\sqrt{33}}{7}, \frac{4}{7}, \pm\frac{\sqrt{33}}{4}, \pm\frac{4}{\sqrt{33}}, \frac{7}{4}, \mp\frac{7}{\sqrt{33}}$. **8.** $\pm\frac{3}{\sqrt{10}}, \mp\frac{1}{\sqrt{10}}, -3, -\frac{1}{3}, \mp\sqrt{10}, \pm\frac{\sqrt{10}}{3}$. **9.** $\pm\frac{5}{\sqrt{29}}, \pm\frac{2}{\sqrt{29}}, \frac{5}{2}, \frac{2}{5}, \pm\frac{\sqrt{29}}{2}, \pm\frac{\sqrt{29}}{5}$. **10.** $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}, -\sqrt{3}, -\frac{2}{\sqrt{3}}, 2$. **11.** $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}, \frac{1}{\sqrt{3}}, -2, -\frac{2}{\sqrt{3}}$. **12.** $\frac{\sqrt{5}}{3}, -\frac{2}{3}, -\frac{\sqrt{5}}{2}, -\frac{2}{\sqrt{5}}, -\frac{3}{2}, \frac{3}{\sqrt{5}}$. **13.** $-\frac{3}{5}, -\frac{4}{5}, \frac{3}{4}, \frac{4}{3}, -\frac{5}{4}, -\frac{5}{3}$. **18.** $45°, 135°, -225°, -315°$; $45°, 135°, 405°, 495°$. **19.** $60°, 240°, -120°, -300°$; $60°, 240°, 420°, 600°$. **20.** $135°, 225°, -135°, -225°$; $135°, 225°, 495°, 585°$. **21.** $150°, 330°, -30°, -210°$; $150°, 330°, 510°, 690°$.

Art. 45. **1.** For $945°$: $-\sin 45°, -\cos 45°, \tan 45°, \cot 45°, -\sec 45°, -\operatorname{cosec} 45°$; for $420°$: $\cos 30°, \sin 30°, \cot 30°, \tan 30°, \operatorname{cosec} 30°, \sec 30°$; etc.

CHAPTER VI.

Art. 46. **8.** $\cos(x+y) = .7874$, $\sin(x+y) = .6164$. (Verify by tables.)

Art. 47. **3.** $\sin(x-y) = -.1582$, $\cos(x-y) = .9874$. (Verify by tables.)

Art. 50. 7. $\cos 6x = \cos^2 3x - \sin^2 3x = 1 - 2\sin^2 3x = 2\cos^2 3x - 1$, $\sin 6x = 2\sin 3x \cos 3x$. 8. $\cos 3x = \cos^2 \frac{3}{2}x - \sin^2 \frac{3}{2}x = 2\cos^2 \frac{3}{2}x - 1 = 1 - 2\sin^2 \frac{3}{2}x$, $\sin 3x = 2\sin \frac{3}{2}x \cos \frac{3}{2}x$. 9. $\sin \frac{3}{4}x = 2\sin \frac{3}{8}x \cos \frac{3}{8}x$, $\cos \frac{3}{4}x = \cos^2 \frac{3}{8}x - \sin^2 \frac{3}{8}x = 1 - 2\sin^2 \frac{3}{8}x = 2\cos^2 \frac{3}{8}x - 1$. 10. $\cos 6x = \sqrt{\frac{1 + \cos 12x}{2}}$, $\sin 6x = \sqrt{\frac{1 - \cos 12x}{2}}$. 11. $\cos 3x = \sqrt{\frac{1+\cos 6x}{2}}$, $\sin 3x \sqrt{\frac{1 - \cos 6x}{2}}$. 12. $\sin \frac{3}{4}x = \frac{1}{2}\sqrt{2(1-\cos \frac{3}{2}x)}$, $\cos \frac{3}{4}x = \frac{1}{2}\sqrt{2(1+\cos \frac{3}{2}x)}$.

Art. 52. 28. a. .9501, − .2047, .3122, .9788, .8660, .9938, .5, .1111, 1.7321, 8.9446, 3.0433, − .2090. b. .2047, − .9501, − .9788, − .3122, .8660, − .9938, .5, .1111, 1.7321, − 8.9446, − .2090, 3.0433. c. − .2047, .9501, − .9788, − .3122, − .8660, .9938, .5, .1111, − 1.7321, 8.9446, .2090, − 3.0433. d. − .9501, .2047, .3122, .9788, − .8660, − .9938, .5, .1111, − 1.7321, − 8.9446, − 3.0433, .2090. 29. a. .6006, − .2392, .7995, .9710, .3919, .7745, .9210, .6326, .4260, 1.2242, .7513, − .2462. b. .2392, − .6006, − .9710, − .7995, .3919, − .7745, .9210, .6326, .4260, − 1.2242, − .2462, .7513. c. − .2392, .6006, − .9710, − .7995, − .3919, .7745, .9210, .6326, − .4260, 1.2242, .2462, − .7513. d. − .6006, .2392, .7995, .9710, − .3919, − .7745, .9210, .6326, − .4260, − 1.2242, − .7513, .2462. 30. − .9805, .0862, − .1963, − .9963, .9600, .9938, .2800, .1111, 3.4287, 8.9446, 4.9955, − .0865; − .0862, .9805, .9963, .1963, .9600, − .9938, .2800, .1111, 3.4287, − 8.9446, − .0865, 4.9955; .0862, − .9805, .9963, .1963, − .9600, .9938, .2800, .1111, − 3.4287, 8.9446, .0865, − 4.9955; .9805, − .0862, − .1963, − .9963, − .9600, − .9938, .2800, .1111, − 3.4287, − 8.9446, − 4.9955, .0865.

CHAPTER VII.

Art. 55. 2. $b = 70.8$, $a = 56.1$. 3. 7.98 ft. 4. $b = 185$, $c = 192$. 5. $b = 8.237$, $c = 5.464$.

Art. 56. 2. $B = 36° 18'.4$ or $143° 41'.6$, $c = 52.71$ or 5.98. 3. Triangle impossible. 4. $B = 46° 28'$, $C = 68° 32'$, $c = 30.8$ in. 5. $A = 48° 10'$, $C = 108° 25'$, $c = 19.1$ ft.; or $A = 131° 50'$, $C = 24° 45'$, $c = 8.4$ ft.

Art. 57. 2. $A = 77° 12'.9$, $B = 43° 30'.1$, $c = 29.97$. 3. $A = 80° 46'.4$, $C = 63° 48'.6$, $b = 19.4$. 4. $R = 33° 3'.3$, $S = 100° 56'.7$, $t = 39.6$ ft. 5. $P = 29° 41'.2$, $Q = 52° 24'.4$, $r = 20$ in.

Art. 58. 2. $P = 19° 12'$, $Q = 61° 13'$, $R = 99° 35'$. 3. $48° 11'.4$, $58° 24'.7$, $73° 23'.9$. 4. $28° 22'$, $49° 43'$, $101° 56'$, nearly. 5. $93° 41'$, $67° 23'$, $18° 56'$.

Art. 60. 2. $C = 66° 47'$, $b = 698.3$, $c = 845$. 3. 600, 421.5. 4. Triangle impossible. 5. $A = 42° 53' 34''$, $C = 77° 5' 54''$, $c = 394.53$. 6. $C = 64° 24'$, $B = 78° 16'$, $b = 749.1$. 7. $B = 46° 52' 10''$, $C = 111° 53' 25''$, $c = 1767.3$; or $B = 133° 7' 50''$, $C = 25° 37' 45''$, $c = 823.8$.

Art. 61. **2.** $B = 64° 9' 3''$, $A = 42° 50' 57''$, $c = 374$. **3.** $P = 132° 18' 27''$, $Q = 14° 34' 24''$, $r = 67.75$. **4.** $A = 109° 15'.5$, $B = 45° 4'.5$, $c = 440.5$. **5.** $A = 88° 2'.6$, $B = 23° 43'$, $c = 8.439$.

Art. 62. **2.** $53° 7'.8$, $59° 29'.4$, $67° 22'.8$. **3.** $A = 29° 17' 16''$, $B = 31° 55' 31''$. **4.** $44° 48' 15''$, $52° 55' 56''$, $82° 15' 49''$. **5.** $R = 25° 12' 32''$, $S = 48° 11' 22''$.

Art. 63. **3.** 444.72 yd. **4.** 1112.8 yd. **5.** 489.29 yd.; 505.3 yd. **6.** 179.28 ft. **7.** 87.88 ft. **8.** 104.08 ft. **9.** 479.3 yd. **10.** Height = 119.6 ft., $AC = 153.1$ ft.; $BC = 571$ ft. **11.** 469.6 ft.

CHAPTER VIII.

Art. 65. $b = a\cos C + c\cos A$, $c = a\cos B + b\cos A$.

Art. 66. **1.** 74973; 44769.2; 781.6; 149691; 15.4. **2.** 278.9; 33600; 324.9; 237023, 1112810 (with seven-place tables). **3.** 264724; 39824.

Art. 67. **1.** 11988 sq. ft.; $46° 13'.8$, $133° 46'.2$; 111.3, 149.1 ft. **2.** 65652; $126° 38'.3$, $53° 21'.7$; 186.5, 438.7. **3.** $72° 35'.6$, $107° 24'.4$; 161.3, 262.7. **4.** $73° 30'.7$, $106° 29'.3$; area = 587637.5 sq. metres (approximately).

CHAPTER IX.

Art. 73. **2.** (*a*) $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, $\frac{3}{4}\pi$, $\frac{7}{6}\pi$, $\frac{5}{3}\pi$, $\frac{11}{6}\pi$, $\frac{3}{2}\pi$, $\frac{5}{4}\pi$, $-\frac{5}{12}\pi$, $\frac{7}{20}\pi$, $\frac{3}{20}\pi$, $-\frac{11}{60}\pi$, $-\frac{3}{8}\pi$; (*b*) .786, 1.048, 2.367, 3.666, 5.238, 5.762, 4.714, 3.929, −1.309, 1.1, .471, −.576, −2.62. **4.** 90°, 60°, 45°, 30°, 36°, 120°, 157° 30′, 1800°, 720°, 540°, 1080°, 450°, 300°; 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{3}{10}\pi$, $-\frac{\pi}{6}$, $-\frac{3}{8}\pi$, $-\frac{19\pi}{2}$, $-\frac{7}{2}\pi$, $-\frac{5}{2}\pi$, $-\frac{11}{2}\pi$, -2π, $-\frac{7}{6}\pi$; $\frac{\pi}{2}$, $\frac{2}{3}\pi$, $\frac{3}{4}\pi$, $\frac{5}{6}\pi$, $\frac{4}{5}\pi$, $\frac{1}{3}\pi$, $\frac{1}{8}\pi$, -9π, -3π, -2π, -5π, $-\frac{3}{4}\pi$, $-\frac{3}{2}\pi$. **5.** $-135°$, $-900°$, $-240°$, $-165°$, $-4500°$. **7.** $28° 38' 52''.4$, $229° 10' 59''.2$, $171° 53' 14''.4$, $19° 5' 54''.9$, $11° 27' 32''.96$, $286° 28' 44''.1$, $572° 57' 28''.1$, $5° 43' 46''.5$. **9.** Sine, cosine, tangent, cotangent, secant, cosecant, respectively are: $\frac{\pi}{6}$: $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$, $\sqrt{3}$, $\frac{2}{\sqrt{3}}$, 2; $\frac{\pi}{4}$: $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1, 1, $\sqrt{2}$, $\sqrt{2}$; π: 0, −1, 0, $-\infty$, −1, ∞; $-\frac{5}{3}\pi$: $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$, $\frac{1}{\sqrt{3}}$, 2, $\frac{2}{\sqrt{3}}$. **10.** **a.** 3.75; **b.** $-\frac{1}{2}$; **c.** 1.5. **11.** The interior angles of the polygons are respectively: $\frac{3}{5}\pi$, $\frac{2}{3}\pi$, $\frac{5}{7}\pi$, $\frac{3}{4}\pi$, $\frac{4}{5}\pi$, $\frac{5}{6}\pi$, $1\frac{1}{5}\pi$; $\frac{2}{3}\pi$, $\frac{1}{3}\pi$, $\frac{2}{7}\pi$, $\frac{1}{4}\pi$, $\frac{1}{5}\pi$, $\frac{1}{6}\pi$, $\frac{2}{15}\pi$; 1.9, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7; 1.3, 1, .05, .9, .8, .6, .5, .4. **12.** (1) $\frac{20^r}{7}$, $163° 42' 8''.3$; (2) $\frac{1^r}{14}$, $4° 5' 33''.2$; (3) $\frac{1^r}{7920}$, $26''$; (4) $\frac{5^r}{160}$, $1° 47' 25''.8$; (5) $\frac{1^r}{1760}$, $1' 57''.2$; (6) $\frac{264^r}{5}$, 3025.2°. **13.** 10, 5, 2.5, $\frac{5}{4}$, 1.25, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{1}{2}$, 20, 30, 25, $26\frac{2}{3}$ in. **14.** 573.1, 286.5, 191, 35.8, 20.5, 4.8,

1146.2, 1719.2, 3438.5, 206186.6, 103092.8, 45829 in. **15.** 10, 40, 70, 80, 120, 5, 3.75, 1.25 in. **16.** .349, 4.363, 8.552, 23.562, 43.633, .0873, .1309, .00145, .00242 in. **17.** 50, 25, 12.5, 8½, 6¼, 4½, 3½, 2½, 100, 150, 125, 33⅓ sq. in.

CHAPTER X.

Art. 83. **4.** 238890 mi. (approximately), 347.5. **5.** About 57′ 2″; about 1 : 13.5. **6.** About 93757000 mi. **7.** 206265 times the distance of the earth from the sun; 3.26 yr. **8.** 9 ft. 2.6 in. **9.** 76 ft. 9.5 in. **10.** 4′ 35″. **11.** 15.708 yd. **12.** 13′ 1″.3.

CHAPTER XI.

Art. 85. **3.** $n\pi + (-1)^n \frac{\pi}{3}$; 60°, 120°, 420°, 480°. **4.** $n\pi + (-1)^{n+1}\frac{\pi}{6}$; 210°, 330°, 570°, 690°. **5.** $n \cdot 180° + (-1)^n 72° 30'$; 72° 30′, 107° 30′, 432° 30′, 467° 30′. **6.** $n \cdot 180° + (-1)^n 23° 25'$; 23° 25′, 156° 35′, 383° 25′, 416° 35′. **7.** $n \cdot 180° + (-1)^n 215° 20'$, or $n \cdot 180° + (-1)^{n+1} 30° 20'$; 215° 20′, 324° 40′, 575° 20′, 684° 40′.

Art. 86. **2.** $2n\pi \pm \frac{\pi}{6}$, $n \cdot 360° \pm 30°$; 30°, 330°, 390°, 690°. **3.** $n \cdot 360° \pm 7° 40'$; 7° 40′, 352° 20′, 367° 40′, 712° 20′. **4.** $n \cdot 360° \pm 62° 10'$; 62° 10′, 297° 50′, 422° 10′, 657° 50′. **5.** $n \cdot 360° \pm 136° 35'$; 136° 35′, 223° 25′, 496° 35′, 583° 25′. **6.** $n \cdot 360° \pm 113° 40'$; 113° 40′, 246° 20′, 473° 40′, 606° 20′.

Art. 87. **2.** $n\pi + \frac{\pi}{3}$, *i.e.* $n \cdot 180° + 60°$; 60°, 240°, 420°, 600°. **3.** $n \cdot 180° + 20° 10'$; 20° 10′, 200° 10′, 380° 10′, 560° 10′. **4.** $n \cdot 180° + 66° 30'$; 66° 30′, 246° 30′, 426° 30′, 606° 30′. **5.** $n \cdot 180° + 35° 40'$; 35° 40′, 215° 40′, 395° 40′, 575° 40′. **6.** $n \cdot 180° + 138°$; 138°, 318°, 498°, 678°. **7.** $n \cdot 180° + 110° 40'$; 110° 40′, 290° 40′, 470° 40′, 650° 40′. **8.** $2n\pi + \alpha$, $n \cdot 360° + A$.

Art. 89. **2.** $\tan^{-1}(-.5)$, *i.e.* $n \cdot 180° + 153° 26'$; $n \cdot 180° + 10° 18'.3$. **3.** $n\pi + \frac{\pi}{2}$. **4.** $n \cdot 180° + 45°$. **5.** $\tan^{-1}(-.25)$, *i.e.* $n \cdot 180° + 165° 57'.8$. **6.** $\tan^{-1}(-2.4)$, *i.e.* $n \cdot 180° + 112° 37'.2$; $\tan^{-1}(-.75)$, *i.e.* $n \cdot 180° + 143° 7'.8$; $\tan^{-1}.67273$, *i.e.* $n \cdot 180° + 33° 55'.8$. **9.** $(2\sqrt{2} + \sqrt{3}) \div 6$. **11.** (*a*) .9730; .2655, −.1066, .4659, .8549; (*b*) ±.9730, ±.1212; ±.2655; ±.0180; ±.1066, ±.6565; ±.4659, ±.8275; ±.8549; ±5.3804. **12.** 43° 5′.5. **13.** 45°.

Art. 90. **5.** $n\pi \pm \frac{\pi}{6}$, $n\pi \pm \frac{\pi}{3}$. **6.** $n\pi + \frac{\pi}{8}$, $n\pi + \frac{5}{8}\pi \left(\textit{i.e.} \frac{n\pi}{2} + \frac{\pi}{8} \right)$. **7.** $n \cdot 180° + 36° 52'.2$. **8.** $n \cdot 180° + 63° 26'$, $n \cdot 180° - 71° 33'.9$. **9.** $n \cdot 180° + 63° 26'$, $(4n - 1) 45°$. **10.** $\frac{n\pi}{4}$, $\frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$. **11.** $n\pi + (-1)^n \frac{\pi}{2}$, $n \cdot 360° - 46° 23'.85$. **12.** $(2n + 1)90°$, $\{4n + (-1)^n\} 15°$. **13.** $\{6n + (-1)^n\} 30°$, $\{10n + (-1)^n\} 18°$, $\{10n - 3(-1)^n\} 18°$. **14.** $n\pi \pm \frac{\pi}{4}$, $n\pi \pm \frac{\pi}{3}$. **15.** $n\pi \pm \frac{\pi}{6}$, $n\pi \pm \frac{\pi}{3}$. **16.** $0, \pm\frac{1}{2}$.

CHAPTER XII.

Art. 94. 9. $\frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$, $\frac{5\tan A - 10\tan^3 A + \tan^5 A}{1 - 10\tan^2 A + 5\tan^4 A}$, $\frac{6\tan A - 20\tan^3 A + 6\tan^5 A}{1 - 15\tan^2 A + 15\tan^4 A - \tan^6 A}$, $\frac{7\tan A - 35\tan^3 A + 21\tan^5 A - \tan^7 A}{1 - 21\tan^2 A + 35\tan^4 A - 7\tan^6 A}$.

PAGES 181–202.

Pages 184, 185. **1.** (*b*) 8° 7′ 40″; 81° 52′ 20″. **3.** (*a*) 70.4 ft.; (*b*) 53° 7′ 42″; 63° 26′ 9″; 63° 26′ 9″; 806.5 yd. **4.** (*a*) 3107.8 yd. (*b*) $4\frac{1}{2}\sqrt{10 - 2\sqrt{5}}$. **5.** 230.03; 385.15. **6.** 220 ft. **7.** 84.2; 35.6; 97.2. **8.** 59.27. **9.** 253.23; 18° 26′.7. **10.** 24.64; 4° 39′.4. **11.** 3° 3′.5. **12.** 46.1 ft.; 99.9 ft. **13.** 3.196 mi. per hour. **14.** 14.11 mi. **15.** 418.4 ft.; 430 ft. **16.** 225 ft. **17.** 106 ft. **18.** 699.85 yd.

Page 187. **15.** (*a*) $\sin A = \frac{1}{n}$, $\operatorname{cosec} A = n$, etc. (*b*) $\cot 2\alpha$, $\sec 2\alpha$. (*c*) − 7.0895. **17.** (*a*) 30°; (*b*) 30°; (*c*) 45°, 60°; (*d*) 1 or $\frac{1}{2}$; (*e*) 60°.

Page 188. **7.** (*b*) 3; (*c*) $1\frac{11}{40}$. **11.** (*a*) $2\sin 5A\cos 3A$; (*b*) (i) $\cos(A+B) + \cos(A-B)$; (*b*) (ii) $\sin 70° + \sin 30°$. **14.** (*a*) 60°; (*b*) 30°, 120°.

Pages 190–192. **11.** (*a*) $B = 46° 53'.3$, $C = 95° 26'.7$; (*b*) 104° 51′.2, 11° 8′.8. **12.** (*a*) 81° 47′.1, 60°, 38° 12′.8; (*b*) 41° 24′.6. **13.** 40° 53′.6, 19° 6′.4, $\sqrt{7} : 2$. **14.** (*a*) 80° 16′.9, 99° 43′.1, 21.2 in., 236.54 sq. in.; (*b*) 77° 9′.7, 139° 20′.5, 84° 15′.5, 59° 14′.6, 149.8. **15.** 78° 27′.3. **16.** S. 70° E., 10.9 mi. per hour. **17.** 1275.6 yd. **18.** 171.08 yd. **19.** (*a*) .4588 mi.; (*b*) .573 distance between the observers (nearly). **20.** 8.5 and 10.8 mi. **21.** 1950.7 ft. **22.** 477 ft. **23.** 4.59 mi. **24.** 4.98 or 3.01 mi. an hour. **25.** 15.03 or 21.99 mi. an hour. **26.** 114.41 ft. **27.** 1034 ft. **30.** 8.57 mi. **31.** 1290 yd. **32.** $h = \frac{a\sin B°\sin A°}{\sin(A° - B°)}$, $d = \frac{a\sin B°\cos A°}{\sin(A° - B°)}$.

Pages 194, 195. **14.** (*a*) 86 ft. (nearly); (*b*) the triangles in which the angles contained by the given sides are 30° 17′.8 and 149° 42′.2 respectively. **16.** 1.765 rd. **3.** (*a*) 12 ft. 9.6 in.; (*b*) $\frac{3}{40}$ and 4° 17′ 50″; (*c*) 12.0031 in.; (*d*) 1.875 and 107° 25′ 46″.5; (*e*) $1\frac{3}{8}$ in., $\frac{8}{10}$ radian, 51° 33′ 58″.3; (*f*) $\frac{11}{120}$, $17\frac{1}{8}$″; (*g*) 20 in., 3 ft., 90.3 yd., 10 in. **4.** 1.9 yd. **6.** (*a*) 1.0872, .9128; (*d*) 30°, 60°, 90°; $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$; $33\frac{1}{3}$, $66\frac{2}{3}$, 100; (*e*) 49.056°, .85619^r; (*f*) 36°, $\frac{\pi}{5}$.

Page 196. **13.** (*a*) 57295.8 ft.; (*b*) 5729 yd. (nearly); (*c*) 32142.85 ft.; (*d*) .1744 in.

Pages 198, 199. **10.** (1) $n\pi \pm \frac{\pi}{6}$; (2) $(2n+1)\frac{\pi}{4}$, or $n\pi \pm \frac{\pi}{4}$;

(3) $2n\pi \pm \frac{\pi}{3}$, or $(2n \pm 1)\pi$; (4) $n\pi + \frac{\pi}{4}$, or $n\pi + \frac{\pi}{6}$; (5) $\sin^{-1}\frac{-1 \pm \sqrt{3}}{2\sqrt{2}}$; (6) $n \cdot 180° + (-1)^n 30°$; (7) $n\pi$, or $n\pi \pm \frac{\pi}{6}$; (8) $n\pi$, or $n\pi + (-1)^n \frac{\pi}{2}$; (9) $2n\pi \pm \frac{\pi}{6}$; (10) $2n\pi \pm \frac{\pi}{3}$; (11) $n\pi \pm \frac{\pi}{4}$; (12) $2n\pi + \frac{\pi}{3}$; (13) $\frac{n\pi}{2} + \frac{\pi}{8}$; (14) $2n\pi + \frac{2\pi}{3}$. **13.** (1) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{4}$, or $\theta = 2n\pi + \frac{\pi}{4}$; (2) $4\theta = n\pi + (-1)^n\theta$, or $\theta = \frac{4n\pi}{5} \pm \frac{\pi}{5}$, or $\theta = \frac{2n\pi}{3}$; (3) $n\pi + (-1)^n \frac{\pi}{10}$, or $n\pi + (-1)^n \frac{13\pi}{10}$; (4) $n\pi \pm \frac{\pi}{6}$, or $n\pi \pm \frac{\pi}{4}$; (5) $\theta = n\pi \pm \frac{\pi}{2}$; (6) $n \cdot 180° \pm 54°40'.6$; (7) $n\pi + \frac{\pi}{4}$; $n\pi \pm \frac{\pi}{4}$, or $2n\pi \pm \frac{2\pi}{3}$; (8) $\theta = 2n\pi$; (9) $\frac{n\pi}{4} \pm \frac{\pi}{8}$, or $\frac{2n\pi}{3} \pm \frac{\pi}{9}$; (10) $\theta = \frac{4n\pi}{3} + \frac{2\pi}{3}$, or $8n\pi \pm \frac{\pi}{12}$; (11) $n\pi \pm \frac{\pi}{4}$; (12) $n\pi + \frac{\pi}{4}$; (13) $n\pi + \frac{\pi}{4}$; (14) $n\pi$, or $n\pi - \frac{\pi}{4}$. **14.** (1) $\frac{1}{2}$; (2) $\frac{1}{2\sqrt{3}}$; (3) $x = \frac{1}{4}$, or -8; (4) 2; (5) $\frac{4}{3}$, or $-\frac{4}{3}$. **16.** $\frac{10}{11}$.

Pages 200–202. **6.** $\frac{24}{85}$. **11.** (*b*) -7.0895. **13.** (*a*) (i) $2n\pi \pm \frac{\pi}{2}$; (ii) $2x = n\pi$, or $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$; (iii) $n\pi + \frac{\pi}{12}$; (*b*) $\sqrt{2}$, $-\frac{1}{3}\sqrt{2}$. **14.** (*b*) $\pm\frac{3}{4}$; (*c*) $\frac{b}{a}$; (*d*) .9725. **18.** (v) 1. **19.** 2.4 ft. **21.** $\frac{13}{12}$, $\frac{\sqrt{13}}{2}$, $\frac{169}{120}$. **23.** 33.072, $17\frac{1}{2}$.

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SPHERICAL TRIGONOMETRY

FOR

COLLEGES AND SECONDARY SCHOOLS

BY

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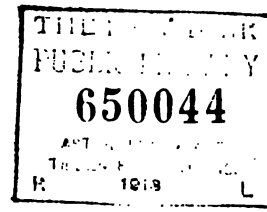

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PREFACE.

THIS book contains little more than what is required for the solution of spherical triangles and related simple practical problems. The articles on spherical geometry are necessary for those who have not already studied that subject; for others, they provide a useful review. More than usual attention has been given to the measurement of solid angles. The explanations in connection with the astronomical problems are somewhat fuller than is customary in elementary text-books on spherical trigonometry.

I am indebted to Mr. W. B. Fite, Ph.B., Fellow in Mathematics at Cornell University, for his kind assistance in reading the proof-sheets; and to Mr. A. T. Bruegel, M.M.E., of the Pratt Institute, Brooklyn, N.Y., for the pleasing character of the diagrams.

D. A. MURRAY.

CORNELL UNIVERSITY,
May, 1900.

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SPHERICAL TRIGONOMETRY.

CHAPTER I.

REVIEW OF SOLID AND SPHERICAL GEOMETRY.

On beginning the study of spherical trigonometry it is advisable to recall to mind or learn some of the definitions and propositions of solid geometry. A clear and vivid conception of the principal properties of the sphere is especially necessary. The definitions and theorems which will be used frequently in the following pages, are quoted in this chapter.*

Planes and Lines in Space. Diedral Angles. Solid Angles.

1. ***a.*** Two planes which are not parallel intersect in a straight line. (Euc. XI. 3.)

b. The angle which one of two planes makes with the other is called a diedral angle. Thus, in Fig. 1, the two planes *BD* and

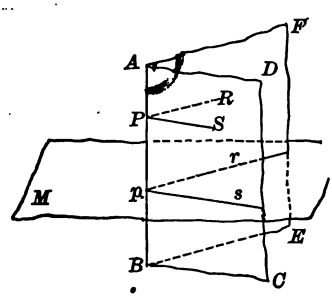

FIG. 1

* As far as possible, references are made to the text of Euclid; since, of the numerous geometrical text-books in English-speaking countries, his work is the one which is most largely used. Those who use a text-book other than Euclid's can substitute the appropriate references.

AE intersect in the straight line *AB*, and form the diedral angle *FABC*.

c. The planes *AE* and *AC* are called the *faces*, and the line *AB* is called the *edge*, of the diedral angle. The faces are unlimited in extent. The magnitude of the diedral angle depends, not upon the extent of its faces, but only upon their relative position. (Just as the magnitude of a plane angle depends, not upon the lengths of its boundary lines, but upon their relative position.)

d. If *PR* be drawn perpendicular to *AB* in the plane *AE*, and *PS* be drawn perpendicular to *AB* in the plane *AC*, the angle *RPS* is called *the plane angle of the diedral angle.*

e. If a plane is drawn perpendicular to the edge of a diedral angle, the intersections of this plane with the faces of the diedral angle form the plane angle of the diedral angle. (See Euc. XI. 4.) Thus, if the plane *M* be passed through *p* perpendicular to *AB*, the intersections, *pr*, *ps*, of the plane *M* and the planes *AE*, *AC*, form the angle *rps* which is the plane angle of *FABC*.

f. All plane angles of the same diedral angle are equal. (See Euc. XI. 10.) Hence, *the plane angle can be taken as the measure of the diedral angle.*

2. *a.* If a straight line be at right angles to a plane, every plane which passes through the line is at right angles to that plane. (Euc. XI. 18.)

b. If two planes which cut one another be each of them perpendicular to a third plane, their common section is perpendicular to the same plane. (Euc. XI. 19.)

3. *a.* When three or more planes meet in a common point, they are said to form a *solid angle*, or a *polyedral angle*, at that point.

The point in which the planes meet is called the *vertex* of the solid angle; the intersections of the planes are called its *edges;* the portions of the planes between the edges are called its *faces;* the plane angles formed by the edges are called its *face angles;* and the diedral angles formed at the edges by the planes are called the *diedral angles* (or the *edge angles*) of the solid angle.

Thus, in Fig. 2, for the solid angle formed at S: the vertex is S; SB, SC, SD, SE, are the edges; BSE, ESD, etc., are the faces; the face angles are the angles BSE, ESD, DSC, CSB; the diedral (or edge) angles are $BESD$, $EDSC$, etc.

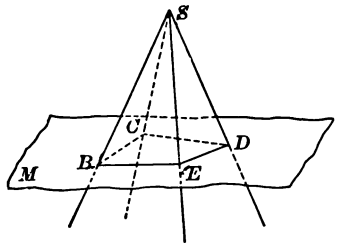

FIG. 2

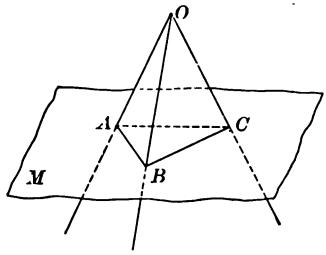

FIG. 3

b. A solid angle with three faces is called a *triedral* angle. Thus, the solid angle at O (Fig. 3) is a triedral angle.

(The measurement of solid angles is discussed in Art. 61. The magnitude of the solid angle in nowise depends upon the lengths of its edges.)

4. ***a.*** The sum of any two face angles of a triedral angle is greater than the third. (See Euc. XI. 20.)

b. The sum of the face angles of any solid angle is less than four right angles (Euc. XI. 21). (This is true, in general, only when the polygon, say $BEDC$ (Fig. 2), formed by the intersections of the faces with a cutting plane M, does not have a reentrant angle; in other words, when the polygon $BEDC$ is convex.

Geometry of the Sphere.

For the benefit of those who have not studied the geometry of the sphere, proofs of a few of its propositions are either outlined, or given in detail. Some propositions can be proved very easily; hence, only their enunciations are given. Other properties of the sphere will be proved when they are required. (See Arts. 53, 54, 57, 62, 65.) The use of a globe on which figures can be drawn, will be of great assistance to the student. If such a globe is not at hand, a terrestrial or celestial globe can afford some service.

5. The sphere and its plane sections.

***a.* Definitions.** A *spherical surface* is a surface all points of which are equidistant from a point called the *centre*. A *sphere* is a solid bounded by a spherical surface. The surface of a sphere can be generated by the revolution of a semicircle about its diameter. A *radius* of a sphere is a straight line joining the centre to any point on the surface. According to the definition of a sphere, all the radii of a sphere are equal. A *diameter* of a sphere is a straight line passing through the centre and terminated at both ends by the surface. A *plane section of a sphere* is a figure whose boundary is the intersection of a plane and the surface of the sphere.

***b.* Proposition.** *The boundary of every plane section of a sphere is a circle.*

Let the sphere whose centre is at O be cut by a plane in the section ABD; then ABD is a circle. Through O draw OC perpendicular to the plane ABD. Let A and B be any two points in the boundary of the section ABD. Draw OA, OB, CA, and CB. In the two triangles OCA and OCB, the angles at C are equal (both being right angles), the side OC is common, and the side OA is equal to the side OB, since both are radii of the sphere. Hence the triangles are equal in every respect, and CA is equal to CB. But A and B are any two points on the boundary of the section; hence all points on the boundary are equidistant from C. Therefore ABD is a circle whose centre is at C, the foot of the perpendicular let fall from the centre O to the cutting plane ABD.

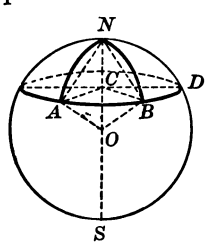

FIG. 4

6. Great and small circles on a sphere.

***a.* Definitions.** The section in which a sphere is cut by a plane is called a *Great Circle* when the plane passes through the centre of the sphere; the section is called a *Small Circle* when the cutting plane does not pass through the centre of the sphere. Thus, on a terrestrial globe the meridians and equator are great circles; the parallels of latitude are small circles. The *Axis* of a circle of

a sphere is the diameter of the sphere perpendicular to the plane of the circle; the extremities of the axis are called the *Poles* of the circle and any of its arcs. Thus, in Fig. 4, Art. 5, N and S are the poles of the circle ABD and of the arcs AB and BD. It is obvious that all circles made by the intersections of parallel planes with a sphere have the same axis and poles. For instance, all parallels of latitude have the same axis and poles, namely, the polar axis of the earth and the North and South Poles.

***b*. Propositions relating to great circles.**

Every great circle bisects the surface of the sphere; *e.g.* the equator bisects the surface of a terrestrial globe.

Any two great circles bisect each other; *e.g.* the meridians bisect one another at the poles. All great circles of a sphere are equal; since their radii are radii of the sphere.

A great circle can be passed through any two points on a sphere; since a plane can be made to pass through these two points and the centre of the sphere, and this plane intersects the surface of a sphere in a great circle. In general, only one great circle can be drawn through two points on a sphere, since these points and the centre determine a plane; but, when the two given points are at the ends of a diameter an infinite number of great circles can be drawn through them; *e.g.* the meridians passing through the North and South Poles.

***c*. Definitions.** By **distance between two points on a sphere** is meant the shorter arc of the great circle passing through them. It is shown in Art. 20 that this arc is the *shortest line* that can be drawn on the surface of the sphere from the one point to the other. For example, the arc NA in Fig. 4 measures the distance between the points N and A. [Ex. Distance between N and S?]

NOTE. The theorem in Art. 20 can be shown *mechanically* by taking two points on a parallel of latitude on a globe and letting a string be stretched taut from one point to the other. The string will not lie on the parallel, but will evidently be in a plane which passes through the centre of the sphere. If the two points be on a meridian, the stretched string will lie on the meridian.

By **angular distance** between two points on a sphere is meant the angle subtended at the centre of the sphere by the arc joining the given points. Thus in Fig. 4 the angle NOA is the angular distance of A from N.

***d.* Propositions and definitions relating to small and great circles.** In Fig. 4 all the arcs of great circles, as *NA, NB, ND,* drawn from points on the circle *ABD* to the pole *N,* are equal. Thus the arcs of meridians on a terrestrial globe drawn from a parallel of latitude to the North Pole are equal. The chords *NA, NB, ND,* are all equal; the angles *AON, BON, DON,* are likewise equal. It thus appears that all points in the circumference of a circle on a sphere are equally distant from a pole of the circle, whether the distance be measured by the arc of a great circle joining one of the points and the pole, or by the straight line joining the point and the pole, or by the angle which such an arc or chord subtends at the centre of the sphere.

Definitions. The last mentioned angle is called the *angular radius* of the circle. The angular radius of a great circle is evidently a right angle. The *polar distance* of a circle on a sphere is its distance from its pole, the distance being measured along an arc of a great circle passing through the pole. Thus the north polar distance of a parallel of latitude is its distance from the North Pole measured along a meridian. The term *quadrant,* when used in connection with a sphere, usually means an arc equal in length to one-fourth of a great circle. The polar distance of each point on a great circle is evidently a quadrant; *e.g.* a point on the equator is at a quadrant's distance from the North or South Pole. Points on a great circle are equidistant from both its poles. The polar distance of a circle may be called the *radius* of the circle.

7. To draw circles upon the surface of a sphere about a given point as pole.

(*a*) *With a pair of compasses.* Open the compasses until the distance between the points of the compasses is equal to the chord of the polar distance (or, what is the same thing, the chord subtended by the angular radius) of the required circle. Then, one point being placed and kept fixed at the pole, the other can describe the circle.

(*b*) *With a string.* Take a string equal in length to the polar distance of the required circle. If the string be kept stretched

taut, and one end be fixed at the pole while the other end moves on the sphere, the required circle will be described.

In order to describe a great circle the polar distance must be taken equal to a quadrant of the sphere.

8. Proposition. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points, it is the pole of the great circle passing through these points.*

If the point P be at a quadrant's distance from each of the points A and B, then P is the pole of the great circle passing through A and B. Let O be the centre of the sphere, and draw OA, OB, OP. Since PA and PB are quadrants, the angles POA and POB are right angles. Hence PO is perpendicular to the plane AOB (Euc. XI. 4); therefore P is the pole of the great circle ABL.

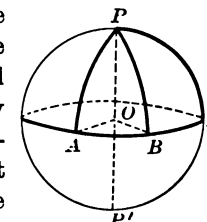

FIG. 5

9. Problem. *Through two given points to draw an arc of a great circle.* About each point as a pole draw a great circle (Art. 7). The two points of intersection of the great circles thus drawn are each at a quadrant's distance from the two given points; and hence, by Art. 8, are the poles of the great circle through the two given points. Accordingly, the required arc will be obtained by describing a great circle about either of these poles.

NOTE. If the two given points are diametrically opposite, an infinite number of great circles can be drawn through them. (Art. 6. *b*.)

10. Lines and planes which are tangent to a sphere.

***a*. Definitions.** A straight line or a plane is said to be *tangent* to a sphere when it has but one point in common with the surface of the sphere. The common point is called the *point of contact* or *point of tangency*.

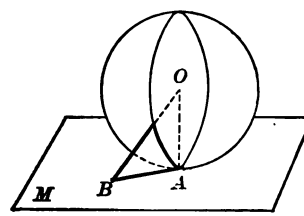

FIG. 6

***b*. Propositions.** (See Fig. 6.)

A plane or a line perpendicular to a radius at its extremity is tangent to the sphere. [Suggestion for proof: The perpendicular is the shortest line that can be drawn from a point to a plane.]

A tangent to an arc of a great circle at any point of the arc is perpendicular to the radius (of the sphere) drawn to the point.

11. On spherical angles.

***a*. Definitions.** *The angle made by any two curves* meeting in a common point is the angle formed by the two tangents to the curves at that point. Thus in Fig. 7, the angle made by the curves C_1 and C_2 at the point P, is the angle T_1PT_2 between the tangents to C_1 and C_2 at P. (This definition applies to all curves, whether they are in the same plane or not.)

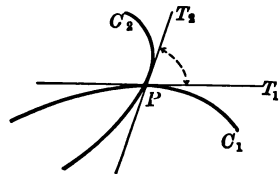

FIG. 7

A spherical angle is the angle formed by two intersecting arcs of great circles on the surface of a sphere. Thus the angle formed by the arcs CA and CB (Fig. 8) is a spherical angle. This angle is the angle ECD between the tangents CE and CD. But ECD is the plane angle of the diedral angle between the planes COA and COB which are the planes of the arcs CA and CB. Thus ***the spherical angle is equal to the diedral angle of the planes of the arcs*** *forming the angle.*

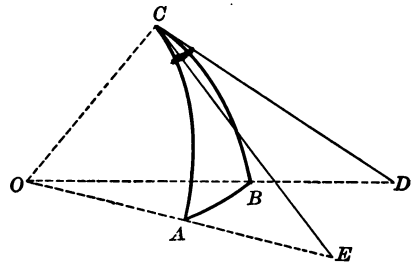

FIG. 8

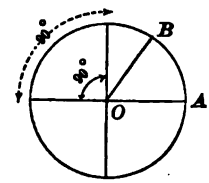

FIG. 9

***b*. Propositions.** (1) If two arcs of great circles intersect, the opposite vertical angles thus formed are equal. Thus in Fig. 49, Art. 57, the angles BAC and $B'AC'$ are equal.

(2) If one arc of a great circle meets another arc of a great circle, the sum of the adjacent spherical angles is equal to two right angles. Thus in Fig. 49, $CAB + CAB' = 2$ right angles.

NOTE. It is shown in plane geometry that angles at the centre of a circle are proportional to their intercepted arcs; hence, the angles can be measured by the arcs. Accordingly, if each right angle at the centre of a circle (Fig. 9) be divided into 90 equal parts called degrees, and the circle be divided into 360 equal parts, also called degrees, then the number of degrees (of angle) in any angle AOB is equal to the number of degrees (of arc) in AB, the arc subtended by AOB. [When it is necessary to distinguish between *degrees of angle* and *degrees of arc*, the former may be called *angular degrees;* and the latter *arcual degrees.*]

c. **Proposition.** *A spherical angle is measured by the arc of a great circle described with its vertex as a pole and included between its boundary arcs, produced if necessary:*

Let ABC and $AB'C$ be two intersecting arcs of great circles on the sphere S whose centre is at O. Pass the plane BOB' through O perpendicular to AC, and let this plane intersect the planes ABC and $AB'C$ in the radii OB and OB', and intersect the sphere in the great circle $B'BL$. From the construction, A is the pole of the great circle $B'BL$. By Art. 1. *e.* BOB' is the plane angle of the diedral angle $BACB'$, and, accordingly (Art. 11. *a*), is equal to the spherical angle BAB'. Now, by the preceding note, the number of degrees in the arc BB' is equal to the number of degrees in the angle BOB'. Hence, the number of degrees in the arc BB' is equal to the number of degrees in the angle BAB'. In other words, the spherical angle BAB' is measured by the arc BB' of which A is the pole.

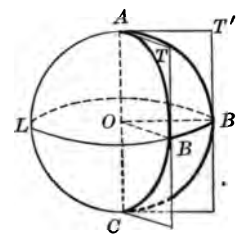

FIG. 10

This can be illustrated on a terrestrial globe. For instance, the angle at the North Pole between the meridians of Paris and New York is 76° 2′ 25.5″; and this is the number of degrees of arc intercepted by these meridians on the equator.

d. The great circles drawn through any point on a sphere are perpendicular to the great circle of which the point is the pole.

For instance, the meridians of longitude cross the equator at right angles.

e. The distance of any point on the surface of a sphere, from a circle traced thereon, is measured by the shorter arc of a great circle passing through the point and perpendicular to the given circle; that is, by the shorter arc of the great circle passing through the given point and the pole of the given circle. For example, on a globe the latitude of any place (*i.e.* its distance in degrees from the equator) is measured by the arc of the meridian intercepted between the place and the equator.

N.B. When an *arc on a sphere* is referred to, an arc of a great circle is meant, unless expressly stated otherwise.

ON SPHERICAL TRIANGLES.

12. Definitions. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles. The bounding arcs are the *sides* of the polygon; the points of intersection of the sides are the *vertices* of the polygon, and the angles which the sides make with one another are the *angles* of the polygon. A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

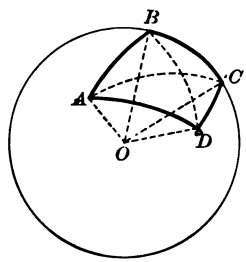

Fig. 11

A *spherical triangle* is a spherical polygon of three sides.

Thus, in Fig. 11, *ABCD* is a spherical polygon; its sides are *AB, BC, CD, DA*; its angles are *ABC, BCD, CDA, DAB*; its diagonals are *BD* and *AC*; *ADC* and *ABC* are spherical triangles. Since the sides of a spherical polygon are arcs of great circles, their magnitudes are expressed in degrees.* The lengths of the sides can be calculated in terms of *linear* units when the radius of the sphere is known.

A spherical triangle is *right-angled, oblique, scalene, isosceles,* or *equilateral,* in the same cases as a plane triangle. The notation

* The reason for expressing the sides of spherical polygons in degrees is considered more fully in Art. 14.

adopted in discussing the plane triangle will be used for the spherical triangle; namely, the triangle will be denoted by ABC, and the sides opposite the angles A, B, C, will be denoted by a, b, c, respectively.

Two spherical polygons are **equal** if they can be applied one to the other so as to coincide. They are said to be **symmetrical** when the sides and angles of the one are respectively equal to the sides and angles of the other, but arranged in *the reverse order.*

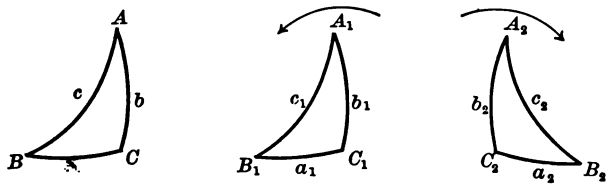

FIG. 12

Thus, the spherical triangles ABC and $A_1B_1C_1$ (Fig. 12) are *equal* if they can be brought into coincidence, say, by sliding one of them, as ABC, over the surface of the sphere until it exactly covers the surface $A_1B_1C_1$. Accordingly, it is evident that if these triangles are equal, the angles A, B, C, are respectively equal to the angles A_1, B_1, C_1, and the sides a, b, c, are respectively equal to the sides a_1, b_1, c_1.* On the other hand, the triangles ABC and $A_2B_2C_2$ are *symmetrical* if the angles A, B, C, are respectively equal to the angles A_2, B_2, C_2, and the sides a, b, c, to the sides a_2, b_2, c_2. In this case, the triangle ABC cannot be brought into coincidence with $A_2B_2C_2$ by a sliding motion over the surface of the sphere.

NOTE 1. Two symmetrical spherical triangles can be brought into coincidence if the surface be covered very thinly with some flexible material. For then ABC can be lifted up, turned over, and the surface bent (or made to 'spring back') in the opposite direction; after this treatment, ABC can be made to coincide with $A_2B_2C_2$.

NOTE 2. The meaning of the phrase *reverse order* can be seen clearly on considering the triangles $A_1B_1C_1$ and $A_2B_2C_2$ above. In $A_1B_1C_1$, on

* Some of the sets of *minimum* conditions necessary for equality of spherical triangles are stated in Art. 13.

going from A_1 to B_1, thence to C_1, and thence to A_1, one goes around any point within the triangle in a *counter-clockwise* direction. In $A_2B_2C_2$, on the other hand, on taking the respective equal angles in the same order as before, that is, on going from A_2 to B_2, thence to C_2, and thence to A_2, one goes round any point within the triangle $A_2B_2C_2$ in a *clockwise* direction. The directions are indicated by the arrows.

13. Propositions. (1) *Two spherical triangles which are on the same sphere, or on equal spheres, and whose parts are in the same order* (as ABC and $A_1B_1C_1$, Fig. 12) *are* **equal** *under the same conditions as plane triangles,* viz.:

(a) When two sides and their included angle in the one triangle are respectively equal to two sides and their included angle in the other;

(b) When a side and its two adjacent angles in the one triangle are respectively equal to a side and its two adjacent angles in the other;

(c) When the three sides of the one triangle are respectively equal to the three sides of the other.

[SUGGESTION FOR PROOFS. Equality can be shown by the same methods as in plane geometry.]

(2) *Two spherical triangles which are on the same sphere, or on equal spheres, and whose parts are in the reverse order* (as ABC and $A_2B_2C_2$, Fig. 12), are **symmetrical** under the conditions (a), (b), (c), above.

[SUGGESTIONS FOR PROOF. Construct* a triangle $A_1B_1C_1$ which is symmetrical to $A_2B_2C_2$. Under the given conditions, according to the preceding proposition, ABC and $A_1B_1C_1$ have all their parts respectively equal, and hence ABC and $A_2B_2C_2$ have all their parts respectively equal, and are accordingly symmetrical.]

On a plane two triangles may have three angles of the one respectively equal to three angles of the other and yet not be equal. On the other hand, as will be made apparent in Arts. 16, 24:

(3) *On the same sphere, or on equal spheres, two triangles which have three angles of the one respectively equal to three angles of the other, are either equal or symmetrical.*

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* For the construction of spherical triangles under various conditions, see Art. 24.

14. Correspondence between the face angles and the diedral angles of a triedral angle on the one hand, and the sides and angles of a spherical triangle on the other.

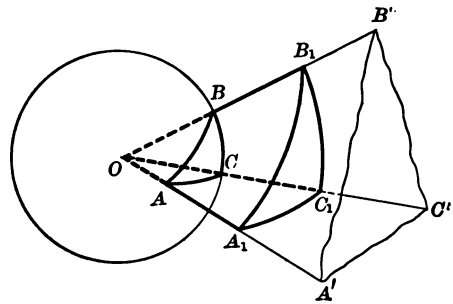

FIG. 13

Take any triedral angle O–$A'B'C'$; let a sphere of any radius, OA say, be described about O as centre; and let the intersections of this sphere with the faces $OA'B'$, $OB'C'$, and $OC'A'$, be the arcs AB, BC, and CA respectively. The sides of the spherical triangle ABC, namely, AB, BC, CA, measure the face angles, AOB, BOC, COA, of the solid angle O–$A'B'C'$ (Art. 11. *b*, Note). By Art. 11 the angles CAB, ABC, BCA, of the spherical triangle ABC are the diedral angles between the planes of the sides, that is, the diedral angles of the solid angle O–$A'B'C'$.

Hence, to find the relations existing between the face angles and the edge angles of a triedral angle, is the same thing as to find the relations between the sides and angles of the spherical triangle, intercepted by the faces, upon the surface of any sphere whose centre is at the vertex of the triedral angle.

NOTE 1. The number of degrees in the intercepted arcs does not depend upon the radius of the sphere. Thus, in Fig. 13, if a sphere is described with a radius OA_1, about O as a centre, the number of degrees in the intercepted arc A_1B_1 is the same as the number of degrees in the intercepted arc AB, for each number is the same as the number of degrees in the angle $A'OB'$.

Since the face angles and diedral angles of a triedral angle are not altered by varying the radius of the sphere, the relations between the sides and angles of the corresponding spherical triangle are independent of the length of the radius.

NOTE 2. Since the side of a spherical triangle measures the angle subtended by it at the centre, the side is measured in degrees or radians. (See Art. 12.) By "sin AB," for example, is meant the sine of the angle AOB, subtended by AB at the centre O.

NOTE 3. A three-sided spherical figure, one or more of whose sides is not an arc of a great circle, is not regarded as a spherical triangle. For example, the figure bounded by an arc of a parallel of latitude and the arcs of two meridians does not correspond to a triedral angle at the centre of the sphere, and is not a spherical triangle as defined in Art. 12.

NOTE 4. A triedral angle, and its corresponding spherical triangle, can be easily constructed. From stiff cardboard cut out a circular sector having any arc between 0° and 360°. On this sector draw any two radii, taking care, however, that no one of the three sectors thus formed shall be greater than the sum of the other two. Along these radii cut the cardboard partly through. Bend the two outer sectors over until their edges meet; a figure like O–ABC (Fig. 13) will be obtained. (Find what happens if the above precaution in drawing the radii is not taken.)

This perfect correspondence between the sides and angles of a spherical triangle on the one hand, and the face angles and dihedral angles of the solid angle subtended at the centre of the sphere by the triangle on the other hand, **is very important,** both for the deduction of the relations between these sides and angles and for the solution of practical problems. This correspondence holds in the case of any spherical polygon and the solid angle subtended by it at the centre of the sphere. (The student may inspect Fig. 11.) Hence, *from any property of polyedral angles an analogous property of spherical polygons can be inferred, and vice versa.*

15. Propositions. (1) *Any side of a spherical triangle is less than the sum of the other two sides.* This follows from Arts. 14 and 4. *a.*

COR. Any side of a spherical polygon is less than the sum of the remaining sides.

(2) *The sum of the sides of a spherical polygon (not re-entrant) is less than 360°.* In other words: *The perimeter of any (non-re-entrant) spherical polygon is less than the length of a great circle.* This important proposition follows from Arts. 14 and 4. *b.*

(3) In an isosceles spherical triangle the angles opposite the equal sides are equal.

(4) The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base is perpendicular to the base, and bisects the vertical angle.

(5) If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.

Cor. If two edge angles of a triedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater diedral angle.

(6) If two sides of a spherical triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.

Ex. Give the corresponding proposition for a triedral angle.

Propositions (3)–(6) can be proved in the same way as the corresponding propositions in plane geometry.

ON POLAR TRIANGLES.

16. *a*. Note. Three straight lines on a plane, no two of which are parallel, intersect in three points, and form one triangle. Three great circles on a sphere have six points of intersection, and form eight spherical triangles. Thus, on a globe, the equator and any two great circles through the poles have as intersections the two poles and the four points where the two great circles cross the equator; and there are eight triangles formed, namely, four in the northern hemisphere and four in the southern.

***b*. Definitions.** If great circles be described with the vertices of a spherical triangle, say ABC (Fig. 14), as poles; and if there be taken that intersection of the circles described with B and C as poles which lies on the same side of BC as does A, namely A_1; and similarly for the other intersections; then a spherical triangle is formed, which is called **the polar triangle** of the first triangle ABC.

Two spherical polygons are *mutually equilateral* when the sides of the one are respectively equal to the sides of the other, whether taken in the same or in the reverse order; the polygons

are *mutually equiangular* when the angles of the one are respectively equal to the angles of the other, whether taken in the same or in the reverse order.

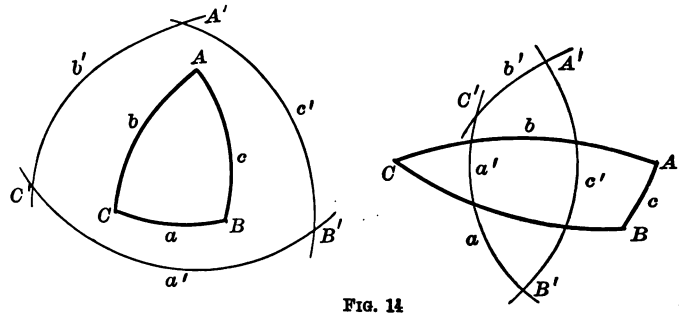

FIG. 14

c. Proposition. *If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.*

If $A'B'C'$ (Fig. 14) is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$. Since A is the pole of the arc $B'C'$, the point A is a quadrant's distance from B'. Also, since C is the pole of $B'A'$, the point C is a quadrant's distance from B'. Since B' is thus a quadrant's distance from both A and C, it is the pole of the arc AC (Art. 8). Similarly it can be shown that A' is the pole of the arc BC, and that C' is the pole of the arc AB. Hence ABC is the polar triangle of $A'B'C'$.

d. Proposition. *In two polar triangles, each angle of the one is the supplement of the side opposite to it in the other.*

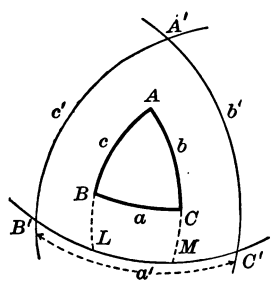

FIG. 15

Let ABC and $A'B'C'$ (Fig. 15) be a pair of polar triangles, in which A, B, C, A', B', C', are the angles, and a, b, c, a', b', c', are the sides. Then

$$A = 180 - a', \qquad A' = 180 - a,$$
$$B = 180 - b', \qquad B' = 180 - b,$$
$$C = 180 - c', \qquad C' = 180 - c$$

Produce the arcs AB and AC to meet $B'C'$ in L and M respectively.

Since B' is the pole of ACM, $B'M = 90°$; and

since C' is the pole of ABL, $LC' = 90°$.

Hence $B'M + LC' = 180°$;

that is, $B'M + MC' + LM = 180°$,

or $B'C' + LM = 180°$. (1)

Since A is the pole of the arc $B'C'$, the arc LM measures the angle A (Art. 11. *c*).

Hence, (1) becomes $A + a' = 180°$, or $A = 180° - a'$.

The other relations can be proved in a similar manner.

Cor. If two spherical triangles are mutually equiangular, their polar triangles are mutually equilateral. If two spherical triangles are mutually equilateral, their polar triangles are mutually equiangular.

Note. On account of the properties in (*d*), a triangle and its polar are sometimes called *supplemental triangles.*

e. **The use of the polar triangle.** Because of the fact that the sides and angles of a triangle are respectively supplementary to the angles and sides of its polar triangle, many relations can be easily derived by reference to the polar triangle. For, if a relation is true for spherical triangles in general, then it is true for the polar of any triangle. Let the relation be stated for the polar triangle; in this statement express the values of the sides and angles of the polar triangle in terms of the angles and sides of the original triangle; the statement thus derived expresses a new relation between the parts of the original triangle. This will be exemplified in later articles.

17. Proposition. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Let ABC be any spherical triangle; it is required to show that

$$180° < A + B + C < 540°.$$

Construct the polar triangle $A'B'C'$. Then, by Art. 16. *d*,

$$A + a' = 180°,\ B + b' = 180°,\ C + c' = 180°.$$

Hence, on adding, $A + B + C + a' + b' + c' = 540°$,

or, $$A + B + C = 540° - (a' + b' + c').$$

Now [Art. 15 (2)] $a' + b' + c'$ is less than 360°, and greater than 0°.

$\therefore (A + B + C) = 540° -$ (something less than 360° and greater than 0°).

$$\therefore A + B + C > 540° - 360°, \textit{ i.e. } A + B + C > 180°;$$

and $$A + B + C < 540° - 0°, \textit{ i.e. } A + B + C < 540°.$$

18. Definitions. ***a.*** The amount by which the sum of the three angles of a spherical triangle is greater than 180° is called its **spherical excess.** It is shown in Art. 57 that the area of a triangle depends upon its spherical excess.

b. A spherical triangle may have two right angles, three right angles, two obtuse angles, or three obtuse angles. For example, on a globe the spherical triangle bounded by any arc (not 90°) on the equator and the arcs of the meridians joining the extremities of the former arc to the North Pole, has two right angles; if the arc on the equator is a quadrant, the triangle has three right angles. The polar of the triangle whose sides are 35°, 25°, 15°, has three obtuse angles. A spherical triangle having two right angles is called a *bi-rectangular triangle*, and a spherical triangle having three right angles is called a *tri-rectangular* triangle. A triangle having one side equal to a quadrant is called a *quadrantal* triangle; one having two sides each a quadrant is said to be *bi-quadrantal*, and one having each of its three sides equal to a quadrant is said to be *tri-quadrantal.*

c. **A lune** is a spherical surface bounded by the halves of two great circles. The *angle of the lune* is the angle made by the two great circles. For instance, on a globe the surface between the meridians 10° W. and 40° W. is a lune; the angle of this lune is equal to 30°. On the same circle or on equal circles lunes having equal angles are equal. (For they can evidently be made to coincide.)

19. On the convention that each side of a spherical triangle be less than 180°. In spherical geometry and trigonometry it is found convenient to restrict attention to triangles the sides of which are each less than a semicircle or 180°. (This convention can be set aside when it is necessary to consider what is called the *general spherical triangle*, in which an element may have any value from 0° to 360°.) A triangle such as $ADBC$ (Fig. 16) which has a side ADB greater than 180°, need not be considered; for its parts can be immediately deduced from the parts of ACB, each of whose sides is less than 180°. It is easily proved that if an angle of a spherical triangle is greater than 180°, the opposite side is also greater than 180°, and *vicê versâ*. Thus, in the triangle $ADBC$, if the angle ACB is greater than 180°, so is the side ADB; and if ADB is greater than 180°, so is the opposite angle. [*Suggestion for proof:* Produce the arc AC to meet the arc ADB.]

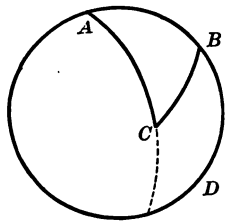

FIG. 16

20. Proposition. The shortest line *that can be drawn on the surface of a sphere between two given points is the arc of a great circle, not greater than a semicircle, which joins the points.*

Let A and B be any two points on a sphere, and let ACB be a great-circle arc not greater than a semicircle; then ACB is the shortest line that can be drawn from A to B on the sphere.

About A as a pole describe a circle DCE with radius AC, and about B as a pole describe a circle FCG with radius BC. It will be shown (1) that C is the only point which is common to both these circles; (2) that the shortest line that can be drawn from A to B on the surface must pass through C.

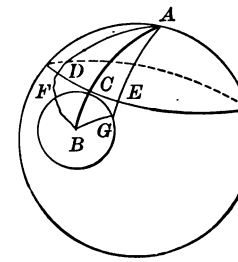

FIG. 17

(1) Take any point G, other than C, on the circle FCG. Draw the great-circle arcs AEG and BG. By Art. 15 (1),

$$AG + GB > AB; \text{ i.e. } AG + GB > AC + CB.$$

Now $AE = AC$, and $GB = CB$.

Hence $AE + GB = AC + CB$:

and, accordingly, $AG > AE$.

Therefore G is outside of the circle DCE. But G is any point (other than C) on the circle FCG. Hence C is the only point common to the circles DCE and FCG.

(2) Let $ADFB$ be any line drawn on the surface from A to B, but not passing through C. Whatever the character of the line AD may be, a line exactly like it can be drawn from A to C; and a line like BF can be drawn from B to C.

[This can be seen by regarding A-DCE as a cap fitting closely to the sphere, and supposing that this cap revolves about A until D is at C. Then a line exactly like AD is drawn from A to C.]

These lines being drawn, there will be a line from A to B which is less than $ADFB$ by the part DF. It has thus been proved that a line can be drawn from A to B through C which is shorter than any other line from A to B which does not pass through C. But C is *any* point on the great-circle arc from A to B. Hence the shortest line from A to B must pass through *every* point in ACB, and, accordingly, must be the arc ACB itself.

NOTE. This proposition can also be proved by the *method of limits*. It is shown that the length of any arc on a sphere is equal to the limit of the sum of the lengths of an infinite number of infinitesimal great-circle arcs inscribed in the given arc. (See Rouché et De Comberousse, *Traité de Géométrie*.) See Art. 6. *c*.

PROBLEMS OF CONSTRUCTION.

21. The actual making of the following constructions will add much to the clearness and vividness of the notions of most students about the surface of a sphere. An easy familiarity with the problems of Arts. 23, 24, which discuss the construction of triangles, will place the student in an advantageous position with respect to spherical trigonometry. This position is similar to that occupied by him, through his knowledge of the construction of plane triangles, when he entered upon the study of plane trigonometry. (See *Plane Trigonometry*, p. 20, Note, Art. 21, Art. 34 (to Case I.), Art. 53.)

N.B. The student should try to make these constructions for himself, and should fall back upon the book only as a last resort.

22. Problems on great circles.

(1) *To find the poles of a given great circle.* About any two points of the given circle as poles, describe great circles; their intersections will be the poles required (Art. 8).

(2) *To draw a great circle through two given points.* About the two given points as poles, describe great circles; about either of the intersections of these circles as a pole, describe a great circle; this will be the circle required. (See Arts. 8, 9.)

(3) *To cut from a great circle an arc $n°$ long.* Separate the points of the compasses by a distance equal to a chord which subtends a central angle of $n°$ in a circle whose radius is equal to the radius of the sphere; place the points of the compass on the great circle; the intercepted arc will be the one required.

(4) *To draw a great circle through a given point perpendicular to a given great circle.* Find a pole of the given circle by (1); draw a great circle through this pole and the given point by (2); this circle will be the one required (Art. 11. *d*).

(5) *To construct a great circle making a given angle with a given great circle, the point of intersection being given.* About the given point of intersection as pole, describe a great circle; on this circle lay off an arc, measured from the given circle, having as many (arcual) degrees as there are (angular) degrees in the given angle; draw a great circle through the extremity of this arc and the given point of intersection; this will be the circle required (Art. 11. *c*).

(6) *To construct a great circle passing through a given point and making a given angle with a given great circle.* [When the given point is on the given circle this problem reduces to problem (5).] It is easily shown that the angle between two great circles is equal to the angular distance (Art. 6. *c*) between their poles. Hence, find a pole of the given circle by (1); about this point as pole describe a second circle whose angular radius (Art. 6. *d*) is equal to the given angle; the pole of the required circle must be on this second circle. About the given point as pole describe a great circle; if the required problem is possible, this circle will either touch or intersect the second circle. The points of contact or intersection are the poles of two great circles, each of which will satisfy the given conditions.

Ex. Discuss the case in which the given point is the pole of the given circle.

23. Construction of triangles. The three sides and the three angles of a spherical triangle constitute its *six parts* or *elements*. *If any three of these six parts be known, the triangle can be constructed.* The *construction* belongs to geometry; the *computation* of the three remaining parts, when three parts are given, is an important part of spherical trigonometry. The sets of three parts that can be taken from the six parts of a spherical triangle are as follows:

I. **Three sides.**

II. **Three angles.**

III. **Two sides** and **their included angle.**

IV. **One side** and **the two adjacent angles.**

V. **Two sides** and **the angle opposite one of them.**

VI. **Two angles** and **the side opposite one of them.**

NOTE. There are *four* construction problems in the case of plane triangles (Plane Trig., Art. 53). When three angles of a spherical triangle are given, there is only one spherical triangle (with the triangle symmetrical to it), as will presently appear, which satisfies the given conditions. When three angles of a plane triangle are given, there is an infinite number of triangles, of the same shape, but of different magnitudes, which have angles equal to the three given angles. Cases IV. and VI. above, in which two angles are given, reduce to a single case in plane trigonometry, namely, the case in which one side and two angles are given; since the sum of the three angles of any plane triangle is 180°.

24. To construct a spherical triangle.

I. **Given the three sides.** On any great circle lay off an arc equal to one of the given sides [Art. 22 (3)]. About one extremity of this arc as pole, describe a circle with a radius (arcual) equal to the second of the given sides; about the other extremity of the arc as pole, describe a circle with a radius equal to the third of the given sides. By arcs of great circles join either of the points of intersection of the last two circles to the extremities of the arc first laid off; the triangle thus formed satisfies the given conditions.

EX. 1. Compare the construction in the corresponding case in plane triangles.

Ex. 2. How many triangles are possible when the first arc is laid off? Are these triangles equal or symmetrical?

Ex. 3. Construct ABC: (a) Given $a = 70°$, $b = 65°$, $c = 40°$; (b) Given $a = 120°$, $b = 115°$, $c = 80°$.

Ex. 4. Determine approximately the angles of these triangles. (See Arts. 11. c, 34.)

II. **Given the three angles.** Calculate the sides of the polar triangle (Art. 16. d); construct it by I. above; construct *its* polar (Art. 16. b); the latter triangle is the one required.

Ex. 1. How many triangles can be drawn when one side of the polar triangle is fixed? Are these triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Construct ABC: (a) Given $A = 85°$, $B = 75°$, $C = 55°$; (b) Given $A = 75°$, $B = 105°$, $C = 100°$.

Ex. 4. Determine approximately the sides of these triangles.

III. **Given two sides and their included angle.** Take any point on any great circle; through this point draw a circle making with the first circle an angle equal to the given included angle [Art. 22 (5)]; from the chosen point and on the first circle bounding this angle, lay off an arc equal to one of the given sides; from the same point and on the second circle bounding the angle, lay off an arc equal to the other given side. Join the extremities of these arcs by the arc of a great circle; the triangle thus formed is the one required.

Ex. 1. How many triangles can be made when the first circle and the point are chosen? Are these possible triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Construct ABC: (a) Given $a = 75°$, $b = 120°$, $C = 65°$; (b) Given $b = 35°$, $c = 70°$, $A = 110°$.

Ex. 4. Determine approximately the remaining parts of these triangles.

IV. **Given a side and its two adjacent angles.**

Either: a. On any arc of a great circle lay off an arc equal to the given side; its extremities will be taken as two vertices of the required triangle. Through one extremity of this arc draw a great circle making with the arc an angle equal to one of the given angles; through the other extremity of the arc draw a

great circle making with the arc (and on the same side as the angle first constructed) an angle equal to the other of the given angles. The point of intersection of these two circles which is on the same side of the arc as the two angles, is the third vertex of the required triangle.

Or: ***b.*** Calculate the corresponding parts of the polar triangle; construct it by III.; construct *its* polar; this will be the triangle required.

Ex. 1. How many triangles are possible when the first arc is chosen? Are these triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Solve Problem III. by means of IV. a, and the polar triangle.

Ex. 4. Construct ABC: (a) Given $a = 75°$, $B = 65°$, $C = 110°$; (b) Given $b = 110°$, $A = 40°$, $C = 63°$.

Ex. 5. Determine approximately the remaining parts of these triangles.

V. **Given two sides and the angle opposite to one of them.** [First, review the corresponding case in plane geometry.]

To construct a triangle ABC when a, b, A, are known: Through any point A of a great circle $ALA'A$ draw the semicircle, ACA', making an angle $A'LAC$ equal to the given angle A. From this semicircle cut off an arc AC equal to b. About C as a pole, with an arc equal (in degrees) to the side a, describe a circle. The intersection B of this circle with ALA' will be the third vertex of the required triangle, A and C being the other two vertices.

Four cases arise, viz.:—

(1) When the circle described about C fails to intersect ALA';

(2) When it just reaches to ALA';

(3) When it intersects the semicircle ALA' in but *one* point;

(4) When it intersects the semicircle ALA' in *two* points.

Case (1) is represented in Figs. 18, 22; case (2), in Figs. 19, 23; case (3), in Figs. 20, 24; and case (4), in Figs. 21, 25. In Figs. 18 and 22 the angle A is respectively acute and obtuse; and similarly for each of the other pairs of figures.

NOTE. In Figs. 18–25 AKA' is a great circle in the plane of the paper, and $ALA'A$ is a great circle at right angles to that plane, ALA' being above the surface of the paper, and the dotted AA' being below. In Figs. 18–21,

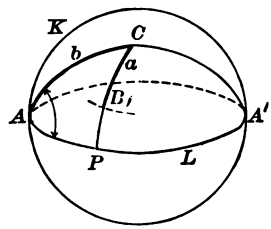

FIG. 18

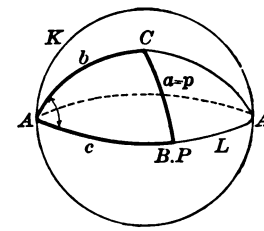

FIG. 19

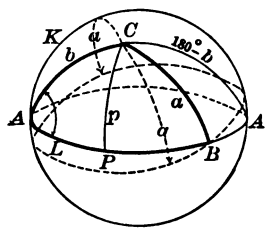

FIG. 20

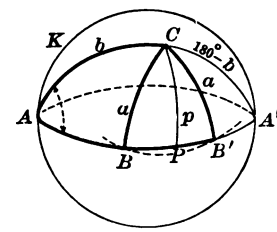

FIG. 21

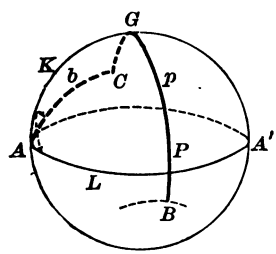

FIG. 22

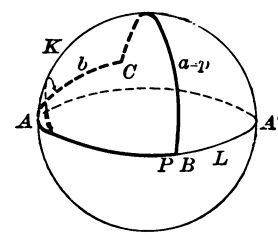

FIG. 23

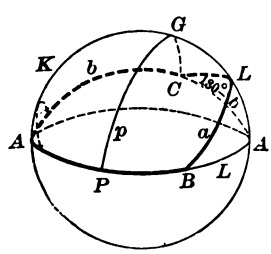

FIG. 24

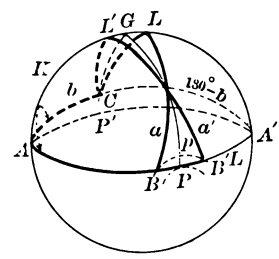

FIG. 25

angle A is acute [equal to $PAK(90°) - KAC$], and the arc AC is in front of the paper. In Figs. 22–25, angle A is obtuse [equal to $PAK(90°) + KAC$), and the arc AC is behind the paper.

In Fig. 21 there are two triangles (not equal or symmetrical) that satisfy the given conditions; and likewise in Fig. 25. Hence V. is **an ambiguous case** in spherical geometry.

In each figure let the perpendicular arc CP be drawn from C to the semicircle ALA', and let its length be called p. [See Ex. 1, p. 101.]

A. *When angle A is acute:*

Fig. 18 shows that the triangle required is impossible, if $CB < CP$, *i.e.* if $a < p$.

Fig. 19 shows that the triangle required is right angled if $CB = CP$, *i.e.* if $a = p$.

Fig. 20 shows that there is but one triangle which satisfies the given conditions, if

$$CB > CP,\ CB > CA, \text{ and } CB < CA';$$

i.e. if $a > p$, $a > b$, and $a < 180° - b$.

Similarly, there is only one triangle if $a > p$, $a < b$, and $a > 180° - b$.

Fig. 21 shows that there are two triangles which satisfy the given conditions, if

$$CB > CP,\ CB < CA, \text{ and } CB < CA';$$

i.e. if $a > p$, $a < b$, and $a < 180° - b$.

B. *When angle A is obtuse:*

Fig. 22 shows that the triangle required is impossible, if $CGB > CGP$, *i.e.* if $a > p$.

Fig. 23 shows that the triangle required is right angled, if $CGB = CGP$, *i.e.* if $a = p$.

Fig. 24 shows that there is but one triangle which satisfies the given conditions, if

$$CLB < CGP,\ CLB < CA \text{ and } CLB > CA';$$

i.e. if $a < p$, $a < b$, and $a > 180° - b$.

Similarly, there is only one triangle if $a < p$, $a > b$, and $a < 180° - b$.

Fig. 25 shows that there are two triangles which satisfy the given conditions, if

$$CLB < CGP,\ CLB > CA, \text{ and } CLB > CA';$$

i.e. if $a < p$, $a > b$, and $a > 180° - b$.

In Fig. 25 produce PGC to meet the great circle $ALA'A$ in P'. Then $CP' = 180° - p$. Since AC and CA' are each greater than CP', it follows that $a > 180° - p$.

It is also apparent from Figs. 20 and 21 that the triangle is impossible,

if A is acute, $a > b$, and $a > 180° - b$;

and it is apparent from Figs. 24 and 25 that the triangle is impossible,

if A is obtuse, $a < b$, and $a < 180° - b$.

Some special cases which may be investigated by the student, are indicated in the exercises on this chapter at page 101.

VI. **Given two angles and the side opposite one of them.** Calculate the corresponding parts of the polar triangle; construct it by V.; construct *its* polar; this is the required triangle. There may be two solutions, since the triangle first constructed may have two solutions.

Ex. 1. Construct ABC: (a) Given $a = 52°$, $b = 71°$, $A = 46°$; (b) Given $a = 99°$, $b = 64°$, $A = 95°$.

Ex. 2. Construct ABC: (a) Given $A = 46°$, $B = 36°$, $a = 42°$; (b) Given $A = 36°$, $B = 46°$, $a = 42°$.

Ex. 3. Determine approximately the remaining parts of these triangles.

N.B. *Questions and exercises on Chapter I. will be found at pages* 101–102.

CHAPTER II.

RIGHT-ANGLED SPHERICAL TRIANGLES.

25. Spherical trigonometry. Spherical trigonometry treats of the relations between the six parts of a triedral angle, or, what is the same thing (Art. 14), the relations between the six parts of the corresponding spherical triangle intercepted on the surface of the sphere. In Art. 24 it has been seen how a triangle can be *constructed* when any three parts are given; Chapters II. and III. are concerned with showing how the remaining parts can be *computed* when any three parts are known. In this chapter the relations between the sides and angles of a right-angled spherical triangle are deduced.* Throughout the book the relations between trigonometric ratios, discussed in plane trigonometry, will be employed.

26. Relations between the sides and angles of a right-angled spherical triangle.

Case I. *The sides about the right angle both less than* 90°.

Let ABC be a spherical triangle which is right angled at C and on a sphere whose centre is at O. First suppose that a and b are

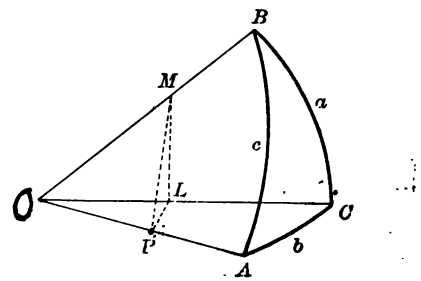

FIG. 26

* These relations can also be obtained from the relations, derived in Chapter III., between the parts of any spherical triangle or triedral angle.

each less than 90°. (It is easily shown, geometrically, that c is then less than 90°.) Draw OA, OB, OC. Take any point P on OA; in the plane OAC draw PL at right angles to OA, and let it meet OC in L; in the plane OAB draw PM at right angles to OA, and let it meet OB in M; and draw ML. Since PL and PM are perpendicular to the line OA, the line OA is perpendicular to the plane LPM (Euc. XI. 4); and, therefore, the plane LPM is perpendicular to the plane OAC (Euc. XI. 18). Also, the plane OCB is perpendicular to the plane OAC, since C is a right angle. Hence, LM, the intersection of the planes LPM and OCB, is perpendicular to the plane OAC (Euc. XI. 19); and hence, MLP and MLO are right angles.

In the triangle OPM, the angle OPM being right,

$$\cos POM = \frac{OP}{OM} = \frac{OP}{OL} \cdot \frac{OL}{OM}.$$

Now, $$\text{angle } POM = c, \quad \frac{OP}{OL} = \cos POL = \cos b,$$

and $$\frac{OL}{OM} = \cos LOM = \cos a.$$

$$\therefore \cos c = \cos a \cos b. \tag{1}$$

In the triangle LPM, angle $PLM = 90°$, and angle $LPM = A$;

$$\sin LPM = \frac{LM}{PM} = \frac{\frac{LM}{OM}}{\frac{PM}{OM}} = \frac{\sin LOM}{\sin POM}.$$

$$\therefore \sin A = \frac{\sin a}{\sin c}. \tag{2}$$

Also, $$\cos LPM = \frac{PL}{PM} = \frac{\frac{PL}{OP}}{\frac{PM}{OP}} = \frac{\tan POL}{\tan POM};$$

whence, $$\cos A = \frac{\tan b}{\tan c}. \tag{3}$$

Also, $$\tan LPM = \frac{LM}{PL} = \frac{\frac{LM}{OL}}{\frac{PL}{OL}} = \frac{\tan LOM}{\sin POL};$$

whence, $$\tan A = \frac{\tan a}{\sin b}. \qquad (4)$$

On remarking that A, a, denote an angle and its opposite side, and that b denotes the other side, the relations for angle B corresponding to (2), (3), (4), can be written immediately; viz.:

$$\sin B = \frac{\sin b}{\sin c} \quad (2'); \quad \cos B = \frac{\tan a}{\tan c} \quad (3'); \quad \tan B = \frac{\tan b}{\sin a} \quad (4').$$

These relations for B can also be deduced directly, by taking any point on OB and making a construction similar to that shown in Fig. 26.

Moreover,

$$\sin A = \tan A \cos A = \frac{\tan a}{\sin b} \cdot \frac{\tan b}{\tan c} = \frac{\tan a}{\tan c} \cdot \frac{1}{\cos b}. \quad [\text{By (3), (4).}]$$

$$\therefore \sin A = \frac{\cos B}{\cos b}. \quad [\text{By (3').}] \qquad (5)$$

Similarly, $$\sin B = \frac{\cos A}{\cos a}. \qquad (5')$$

Also, $$\cos c = \cos a \cos b = \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A}. \quad [\text{By (1), (5), (5').}]$$

$$\therefore \cos c = \cot A \cot B. \qquad (6)$$

For convenience of reference, relations (1)–(6) may be grouped together:

$$\cos c = \cos a \cos b. \qquad (1)$$

$$\sin A = \frac{\sin a}{\sin c}. \qquad \sin B = \frac{\sin b}{\sin c}. \qquad (2), (2')$$

$$\cos A = \frac{\tan b}{\tan c}. \qquad \cos B = \frac{\tan a}{\tan c}. \qquad (3), (3')$$

$$\tan A = \frac{\tan a}{\sin b}. \qquad \tan B = \frac{\tan b}{\sin a}. \qquad (4), (4')$$

$$\sin A = \frac{\cos B}{\cos b}. \qquad \sin B = \frac{\cos A}{\cos a}. \qquad (5), (5')$$

$$\cos c = \cot A \cot B. \qquad (6)$$

Case II. *The sides about the right angle both greater than* 90°.

In Fig. 27, C denotes the right angle, and the sides a, b, are each greater than a quadrant.

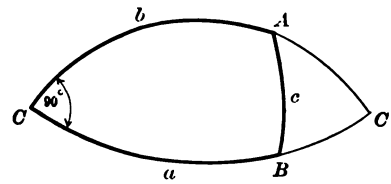

FIG. 27

Form the lune CC' by producing the sides a and b to meet in C'. Then ABC' is a right triangle in which the sides about the right angle are each less than 90°.

$$\therefore \cos c = \cos BC' \cos AC' = \cos(180 - a)\cos(180 - b).$$

Hence $$\cos c = \cos a \cos b.$$

Also,

$$\cos BAC' = \frac{\tan AC'}{\tan AB};\ \textit{i.e.}\ \cos(180° - BAC) = \frac{\tan(180° - AC)}{\tan AB};$$

whence, $$\cos A = \frac{\tan b}{\tan c}.$$

In a similar manner the other relations in (1)–(6) can be shown to be true for ABC (Fig. 27).

NOTE. ABC' is said to be *co-lunar* with ABC. Every triangle has *three* co-lunar triangles, one corresponding to each angle.

Case III. *One of the sides about the right angle less than* 90°, *and the other side greater than* 90°.

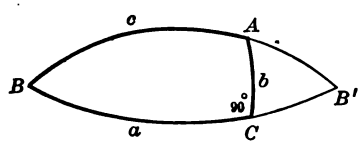

FIG. 28

In ACB let $C = 90°$, $a > 90°$, and $b < 90°$. Complete the lune BB'. Then ACB' is a right-angled triangle in which the sides about the right angle are each less than 90°.

In ACB', $\cos AB' = \cos AC \cos CB'$;

i.e. $\cos(180° - c) = \cos b \cos(180° - a)$;

whence $\cos c = \cos a \cos b$.

Again,

$$\cos CAB' = \frac{\tan AC}{\tan AB'};\ \textit{i.e.}\ \cos(180° - BAC) = \frac{\tan AC}{\tan(180° - BA)};$$

whence, $$\cos A = \frac{\tan b}{\tan c}.$$

In a similar way the other relations in (1)–(6) can be shown to be true for ABC (Fig. 28).

27. On species. Two parts of a spherical triangle are said to be *of the same species* (*or of the same affection*) when both are less than 90°, both greater than 90°, or both equal to 90°. Formula (1), Art. 26, shows that the *hypotenuse* of a right-angled spherical triangle is *less than* 90° when the *sides* about the right angle are *both greater or both less than* 90°; and it shows that the *hypotenuse is greater than* 90° when the *sides are of different species.* Formulas (4) and (4′) show that in a right-angled spherical triangle (since the sines of the sides are positive) *an angle and its opposite side are of the same species.* These important properties can also be deduced geometrically.

EXAMPLES.

N.B. It is advisable to remember the result of **Ex. 1.**

1. State relations (1)–(6), Art. 26, in words.

(1). *cos* hyp. = product of *cosines* of sides.

(6). *cos* hyp. = product of *cotangents* of angles.

(2), (2′). *sin* angle = *sin* opposite side ÷ *sin* hyp.

(3), (3′). *cos* angle = *tan* adjacent side ÷ *tan* hyp.

(4), (4′). *tan* angle = *tan* opposite side ÷ *sin* adjacent side.

(5), (5′). *cos* angle = *cos* opposite side × *sin* other angle.

[Compare (2), (3), (4), with the corresponding formulas in plane triangles.]

2. Deduce formulas (1)–(4) by means of a figure in which P is anywhere on OB (see Fig. 26).

3. Deduce formulas (1)–(4) by means of a figure in which P (see Fig. 26) is: (a) in OA produced; (b) in OB produced; (c) at the point A; (d) at the point B.

4. The two sides about the right angle of a spherical triangle are 60° and 75°; find the hypotenuse and the other angles by means of relations (1), (4), (4′), Art. 26. Check (or test) the result by means of other formulas.

5. In ABC, given $A = 47° 30'$, $B = 53° 40'$, $C = 90°$; find the remaining parts, and check the results.

6. Solve some of the examples in Art. 31, and check the results.

28. Solution of a right-angled triangle.

N.B. The student is advised to investigate this subject independently; and, *before* reading this article, to *put in writing* in an orderly manner his ideas about the solution of right triangles. These ideas will thus be made clearer in his mind, and his subsequent reading will be easier.

In a right triangle there are five elements beside the right angle. These five elements can be taken in groups of three in ten different ways. Each of these ten groups is involved in the ten relations derived in Art. 26; the three elements of each group are, accordingly, connected by one relation.

Ex. (a) Write all the groups of three that can be formed from a, b, c, A, B, such as a, b, c; a, b, A; etc.

(b) Write the relation connecting the elements of each group.

It follows that if any two elements of a right-angled spherical triangle besides the right angle be given, then the remaining three elements can be determined. The method of finding a third element is as follows:

Write the relation involving the two given elements and the required element; solve this equation for the required element.

Check (or *test*). When the required elements are obtained, the results can be checked by examining whether they satisfy relations which have not been employed in the solution, and, preferably, the relation involving the newly found elements.

E.g., suppose that A, b, are known, C being 90°; then a, c, B, are required. Side a can be found by (4); side c, by (3); and angle B, by (5). The values found for a, c, B, can be checked by (3′).

NOTE 1. The cosine, tangent, and co-tangent of sides and angles greater than 90°, are *negative*. *Careful attention must be paid to the algebraic signs of the trigonometric functions appearing in the work.*

NOTE 2. The properties stated in Art. 27 are very useful.

NOTE 3. Determine each element from the given elements alone. If an element is found erroneously and then used in finding a second element, the second element will also be wrong.

The ten possible groups of three elements correspond to the following *six* cases for solution, in which the given elements are respectively:

(1) Two sides.
(2) Hypotenuse and a side.
(3) Hypotenuse and an angle.
(4) Two angles.
(5) Side and opposite angle.
(6) Side and adjacent angle.

Before proceeding to the solution of numerical examples, it is necessary to refer more particularly to one of these cases; and also to call attention to the fact that the ten formulas for right triangles (Art. 26) may be grouped in two very simple and convenient rules.

29. The ambiguous case. When the given parts are a side and its opposite angle, there are two triangles which satisfy the given conditions. For, in ABC (Fig. 29), let $C = 90°$, and let A and CB (equal to a) be the given parts. Then, on completing the lune AA', it is evident that the triangle $A'BC$ also satisfies the given conditions, since $BCA' = 90°$, $A' = A$, and $CB = a$. The remain-

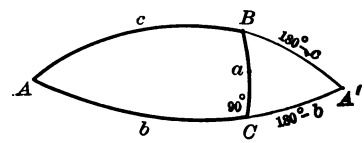

FIG. 29

ing parts in $A'BC$ are respectively supplementary to the remaining parts in ABC; thus $A'B = 180° - c$, $A'C = 180° - b$, $A'BC = 180° - ABC$.

This ambiguity is also apparent from the relations (1)–(6), Art. 26. For, if a, A, are given, then the remaining parts, namely,

c, b, B, are all determined from their sines [see (2), (4), (5'),]; and, accordingly, c, b, B, may each be less or greater than 90°. On the other hand, if, for instance, a and c are given, then b is determined from its cosine by (1); and there is no ambiguity, because b is less or greater than 90° according as cos b is respectively positive or negative.

N.B. Both solutions should be given in the ambiguous case, unless some information is given which serves to indicate the particular solution that is suitable.

30. Napier's rules of circular parts. The ten relations derived in Art. 26 are all included in two statements, which are called *Napier's rules of circular parts*, after the man who first announced them, Napier, the inventor of logarithms.

Let ABC be a triangle right-angled at C. Either draw a right-angled triangle, and mark the sides and angles as in Fig. 31, or draw a circle and mark successive arcs as in Fig. 32, in which b, a, *Co-B*, *Co-c*, *Co-A*, are

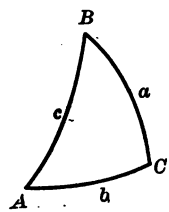

FIG. 30

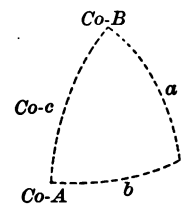

FIG. 31

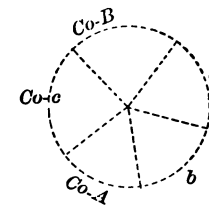

FIG. 32

arranged in the same order as b, a, B, c, A, in Fig. 30. (Here *Co-B*, *Co-c*, *Co-A*, denote the complements of B, c, and A, respectively.) The five quantities, a, b, *Co-B*, *Co-c*, *Co-A*, are known as *circular parts*. That is, the right angle being omitted, the two sides and the complements of the hypotenuse and the other angles are called the circular parts of the triangle.

In Figs. 31 and 32 each part has two circular parts *adjacent* to it, and two circular parts *opposite* to it. Thus, on taking a, for instance, the adjacent parts are b, *Co-B*, and the opposite parts are *Co-c*, *Co-A*. If any three parts be taken, one of them is midway between the other two, and the latter are either its two adjacent parts or its two opposite parts. Thus, if a, b, *Co-A*, be taken, then b is the middle part, and a, *Co-A*, are the adjacent parts; if a, b, *Co-c*, be taken, then *Co-c* is the middle part, and a, b, are opposite parts.

Ex. Take each of the circular parts in turn, write its opposite parts and adjacent parts, thus getting *ten* sets in all.

Napier's rules of circular parts are as follows:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

(The i's, a's, and o's are lettered thus, in order to aid the memory.)

These rules are easily *verified.* For example, on taking a for the middle part,

$$\sin a = \tan b \tan (90° - B) = \tan b \cot B. \qquad \text{[See Art. 26 (4').]}$$

$$\sin a = \cos (90° - A) \cos (90° - c) = \sin A \sin c. \qquad \text{[See Art. 26 (2).]}$$

Again, on taking *Co-A* for the middle part,

$$\sin (90° - A) = \tan b \tan (90° - c), \text{ } i.e. \text{ } \cos A = \tan b \cot c.$$
[See Art. 26 (3).]

$$\sin (90° - A) = \cos a \cos (90° - B), \text{ } i.e. \text{ } \cos A = \cos a \sin B.$$
[See Art. 26 (5').]

In a similar way each of the remaining three parts can be taken in turn for the middle part, and the remaining six relations of Art. 26 shown to agree with Napier's rules.*

NOTE. One may either memorize the relations in Art. 26 (or Ex. 1, Art. 27), or use Napier's rules. Opinions differ as to which is the better thing to do.

Ex. 1. Verify Napier's rules by showing that they include the 10 relations in Art. 26.

Ex. 2. Prove Napier's rules.

31. Numerical problems. In solving right triangles the procedure is as follows:

(1) Indicate the two given parts and the three required parts.

---

* This is only a *verification* of Napier's rules. One *proof* of the rules would consist of the derivation of the relations in Art. 26 *plus* this verification. These rules were first published by Napier in his work *Mirifici Logarithmorum Canonis Descriptio* in 1614. Napier indicated a geometrical method of proof, and deduced the rules as special applications of a more general proposition. They are something more than mere technical aids to the memory. For an explanation of this and of their wider geometrical interpretation, see Charles Hutton, *Course in Mathematics* (edited by T. S. Davies, London, 1843), Vol. II. pp. 24–26; Todhunter, *Spherical Trigonometry,* Art. 68; E. O. Lovett, *Note on Napier's Rules of Circular Parts* (Bulletin Amer. Math. Soc., 2d Series, Vol. IV. No. 10, July, 1898).

(2) Write the relations that will be employed in the solution, and note carefully the algebraic signs of the functions involved. The noting of these signs will serve to show (unless a part is determined from its sine) whether a required part is less or greater than 90°.

(3) For use as a *check*, write the relation involving the three required parts.

(4) Arrange the work as neatly and clearly as possible.

N.B. Attention may be directed to the notes in Art. 28. Also see *Plane Trigonometry*, Art. 27 (particularly p. 45, notes 2, 4–6), and Art. 59, p. 107.

NOTE. The trigonometric function of any angle can be expressed in terms of some trigonometric function of an angle less than 90°. See *Plane Trigonometry*, Art. 45.

EXAMPLES.

1. Solve the triangle ABC, given:

$C = 90°$, Solution*: $c =$

$a = 44° 30'$, $A =$

$b = 71° 40'$ $B =$

Formulas: $\cos c = \cos a \cos b$, *Check:* $\cos c = \cot A \cot B$

$\tan A = \tan a \div \sin b$,

$\tan B = \tan b \div \sin a$.

Logarithmic formulas: $\log \cos c = \log \cos a + \log \cos b$,

[If necessary; see *Pl. Trig.*, Art. 27, Note 6.] $\log \tan A = \log \tan a - \log \sin b$,

$\log \tan B = \log \tan b - \log \sin a$,

$\log \cos c = \log \cot A + \log \cot B$ (*check*).

$\log \sin a = 9.84566 - 10$ $\therefore \log \cos c = 9.35092 - 10$

$\log \cos a = 9.85324 - 10$ $\log \tan A = 0.01504$

$\log \tan a = 9.99242 - 10$ $\log \tan B = 0.63403$

$\log \sin b = 9.97738 - 10$ $\therefore c = 77° 2'.1$.

$\log \cos b = 9.49768 - 10$ $A = 45° 59'.5$.

$\log \tan b = 0.47969$ $B = 76° 55'.5$.

Check: $\therefore \log \cot A = 9.98497 - 10$

$\log \cot B = 9.36595 - 10$

$\therefore \log \cos c = 9.35092 - 10$

* To be filled in.

Note. Spherical triangles, like plane triangles, can also be solved without the use of logarithms. (See *Plane Trigonometry*, examples in Arts. 27, 55–62.)

2. Solve the triangle ABC, given:

$C = 90°$, $\quad$ Solution: $c =$

$A = 57° 40'$, $\quad b =$

$a = 48° 30'$. $\quad B =$

Formulas: $\sin c = \dfrac{\sin a}{\sin A}$, $\quad$ *Check:* $\sin B = \dfrac{\sin b}{\sin c}$.

$\sin b = \dfrac{\tan a}{\tan A}$,

$\sin B = \dfrac{\cos A}{\cos a}$.

$\log \sin a = 9.87446 - 10$ $\quad \therefore \log \sin c = 9.94763 - 10$

$\log \cos a = 9.82126 - 10$ $\quad \log \sin b = 9.85459 - 10$

$\log \tan a = 0.05319$ $\quad \log \sin B = 9.90697 - 10$

$\log \sin A = 9.92683 - 10$ $\quad \therefore c = 62° 25'.4$, or $117° 34'.6$.

$\log \cos A = 9.72823 - 10$ $\quad b = 45° 40'.9$, or $134° 19'.1$.

$\log \tan A = 0.19860$ $\quad B = 53° 49'.3$, or $126° 10'.7$.

The check gives $\log \sin B = 9.90696$.

On combining the results according to the principles of Art. 27, the solutions are:

(1) $c = 62° 25'.4$, $b = 45° 40'.9$, $B = 53° 49'.3$;

(2) $c = 117° 34'.6$, $b = 134° 19'.1$, $B = 126° 10'.7$.

3. Solve Ex. 1 without using logarithms.

4. Given $c = 90°$, $A = 57° 40'$, $a = 108° 30'$. Show both by geometry and trigonometry why the solution is impossible.

5. Solve the triangle ABC, in which $C = 90°$, and *check the results*, given:

(1) $a = 36° 25' 30''$, $b = 85° 40'$; (2) $c = 120° 20' 30''$, $a = 47° 30' 40''$;

(3) $c = 78° 25'$, $A = 36° 42' 30''$; (4) $A = 63° 18'$, $B = 37° 47'$;

(5) $a = 76° 48'$, $A = 82° 36'$; (6) $b = 39° 50' 20''$, $A = 47° 50'$;

(7) $a = 47° 40'$, $A = 30° 43'$; (8) $b = 70° 30' 30''$, $B = 80° 40' 20''$;

(9) $a = 108° 45'$, $B = 37° 42'$; (10) $c = 78° 20'$, $B = 47° 50'$;

(11) $A = 110° 27'$, $B = 74° 36'$; (12) $a = 108° 42'$, $b = 63° 26'$;

(13) $A = 124° 30'$, $b = 25° 40'$; (14) $c = 84° 47'$, $b = 39° 43'$.

32. Solution of isosceles triangles and quadrantal triangles.

Isosceles Triangles. Plane isosceles triangles can be solved by means of right triangles, as shown in *Plane Trigonometry*, Art. 32. A spherical isosceles triangle can be solved in a similar way, namely, by dividing it into two right triangles by an arc drawn from the vertex at right angles to the base. This arc bisects the base and the vertical angle.

Quadrantal Triangles. The polar triangle of a quadrantal triangle (Art. 18) is right-angled (Art. 16. *d*). Hence, a quadrantal triangle may be solved by solving its polar triangle by Arts. 28, 31, and then computing the required parts of the quadrantal triangle by Art. 16. *d*.

EXAMPLES.

1. Solve the triangle ABC, in which A and B are equal, and *check the results*, given:

(1) $a = 54° 20'$, $c = 72° 54'$; (2) $a = 66° 29'$, $A = 50° 17'$;

(3) $a = 54° 30'$, $C = 71°$; (4) $c = 156° 40'$, $C = 187° 46'$.

2. Solve the triangle ABC, given:

(1) $c = 90°$, $C = 67° 12'$, $a = 123° 48' 4''$;

(2) $c = 90°$, $A = 136° 40'$, $B = 105° 47'$.

33.* Solution of oblique spherical triangles. It has been seen (*Plane Trigonometry*, Art. 34) that oblique plane triangles can be solved by means of right triangles. Oblique spherical triangles can also be solved by means of right spherical triangles. Relating to spherical triangles there are *six problems of computation;* these correspond to the six problems of construction discussed in Arts. 23, 24. If any three parts of a triangle are given, the triangle can be constructed and the remaining parts can be computed. The several cases for computation will now be solved with the help of right-angled triangles.†

(In the figures in this article the given parts are indicated by crosses.)

N.B. The student is advised to try to solve Cases II.–VI. *before* reading the text.

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* When time is limited this article may be omitted, or merely glanced over.

† Other methods of solving triangles are shown in Chap. IV.

Case I. Given the three sides. In ABC (Figs. 33, 34) let a, b, c, be given, and A, B, C, be required. From C draw CD at right angles to AB, or AB produced. Let the segments AD and DB be denoted by m and n, respectively. If the direction from A to B is taken as the positive direction along the arc AB, then m is positive in Fig. 33 and negative in Fig. 34, while n is positive in both figures.

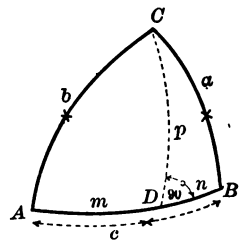

FIG. 33

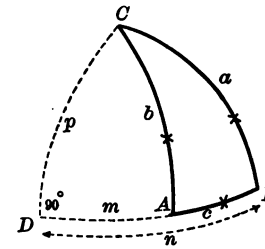

FIG. 34

Special formula. In each figure,

$$\cos a = \cos n \cos p, \text{ and } \cos b = \cos m \cos p.$$

$$\therefore \cos p = \frac{\cos a}{\cos n} = \frac{\cos b}{\cos m}.$$

$$\therefore \frac{\cos n}{\cos m} = \frac{\cos a}{\cos b}.$$

$$\therefore \frac{\cos n - \cos m}{\cos n + \cos m} = \frac{\cos a - \cos b}{\cos a + \cos b}. \quad \text{[Composition and division.]}$$

From this, on applying *Plane Trigonometry*, Art. 52, formulas (7), (8),

$$\tan \tfrac{1}{2}(n+m) \tan \tfrac{1}{2}(n-m) = \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b). \quad (1)$$

Now $n + m = c$; hence, $n - m$ can be found by (1). Then the segments m and n can each be determined. The triangles ADC and BDC can then be solved by Arts. 28, 31; and the solution of ABC can be obtained therefrom.

EX. 1. Solve Exs. 1, 2, Art. 42, by the method outlined above.

EX. 2. Show how to solve this case when the perpendicular is drawn from A to BC.

Case II. Given the three angles. Solve the polar triangle by the method used in Case I.; and therefrom (Art. 16. *d*) compute the parts of the original triangle.

Ex. Solve Exs. 1, 2, Art. 43, by this method.

Case III. Given two sides and their included angle. In ABC (Fig. 35) let a, c, B, be given. Draw AD at right angles to BC, or BC produced.

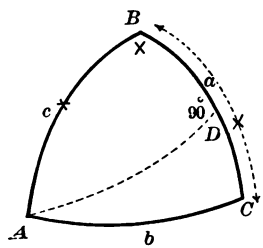

FIG. 35

In ABD, c and B are known; hence, BAD, AD, and BD can be found. In ADC, AD and DC (equal to $a - BD$) are now known; hence DAC, ACD, and AC can be found. Also, $CAB = CAD + DAB$. The student can examine the case in which AD falls outside ABC.

Ex. 1. Show how to solve the triangle by drawing a perpendicular arc from C to AB.

Ex. 2. Solve Exs. 1, 2, Art. 44, by means of right triangles.

Case IV. Given a side and the two adjacent angles. Two methods of solution may be employed.

Either: (1) Solve the polar triangle by the method used in Case III.; and therefrom compute the parts of the original triangle.

Or: (2) In ABC (Fig. 36) let A, B, c, be given. Draw the arc BD at right angles to AC. In ADB, AD, DB, and ABD can be found, since A and AB are known. Now $DBC = ABC - ABD$. In DBC, DB and DBC are now known; hence BC, CD, and C can be found. Then $AC = AD + DC$.

The student can examine the case in which BD falls outside ABC.

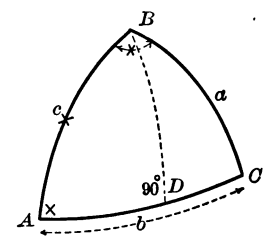

FIG. 36

Ex. 1. Solve the triangle by drawing a different perpendicular.

Ex. 2. How may solution (2) aid in the solution of Case III.?

Ex. 3. Solve Exs. 1, 2, Art. 45, by means of right triangles.

Case V. Given two sides and the angle opposite one of them. (*This may be an ambiguous case;* see Art. 24, V.)

In ABC (Fig. 37) let a, c, A, be given. From B draw the arc BD at right angles to AC to meet AC or AC produced. In ABD, c and A are known; hence AD, DB, and ABD can be found. In DBC, DB and a are now known; hence DBC, C, and DC can be found. Then $AC = AD + DC$, and $ABC = ABD + DBC$.

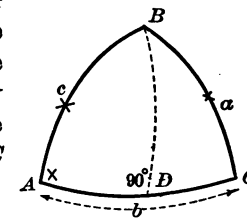

FIG. 37

EX. 1. Examine the cases in which BD falls outside ABC.

EX. 2. Examine the case in which *two* triangles satisfy the given conditions.

EX. 3. Solve Exs. 1, 2, Art. 46, by means of right triangles.

Case VI. Given two angles and the side opposite one of them. Like Case V. this may be ambiguous; see Art. 24, VI. Two methods of solution may be employed.

Either: (1) Solve the polar triangle by the method used in Case V.; and therefrom compute the parts of the original triangle.

Or: (2) In ABC (Fig. 38) let A, C, c, be known. From B draw BD at right angles to AC to meet AC, or AC produced. Solve the triangle ABD; then solve the triangle DBC. The parts of ABC can be computed from these solutions.

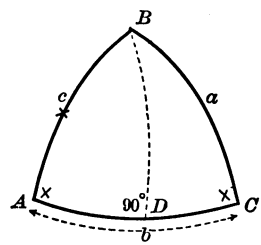

FIG. 38

EX. 1. How may (2) aid in the solution of Case V.?

EX. 2. Solve Exs. 1, 2, Art. 47, by means of right triangles.

EX. 3. Solve the numerical examples in Art. 24.

34. Graphical solution of (oblique and right) spherical triangles. A plane triangle can be solved graphically by drawing to scale a triangle that satisfies the given conditions, and then measuring the required parts directly from the figure (*Plane Trigonometry,*

Arts. 10, 21). A spherical triangle can be solved graphically by drawing (Art. 24) upon any sphere a triangle that satisfies the given conditions, and then measuring the required parts of the triangle. The sides and angles (see Art. 11. *c*) can be measured with a thin, flexible, brass ruler, on which a length equal to a quadrant or a semicircle of the sphere is graduated from 0° to 90° or 180° respectively.

Small slated globes can be obtained fitting into hemispherical cups, whose rims are graduated from 0° to 180° in both directions. With such a globe, cup, and a pair of compasses, the constructions discussed in Art. 24 and the measurements referred to in this article are easily made.

If the student has the means at hand, it is advisable for him to solve some of the numerical problems graphically.

N.B. *Questions and exercises on Chapter II. will be found at p.* 102.

CHAPTER III.

RELATIONS BETWEEN THE SIDES AND ANGLES OF SPHERICAL TRIANGLES.

35. In this chapter some relations between the sides and angles of any spherical triangle (whether right-angled or oblique) will be derived. In the next chapter these relations will be used in the solution of practical numerical problems. The first two general relations (namely, the *Law of Sines* and the *Law of Cosines*), which are by far the most important, can be derived in various ways. In a short course it may be best to deduce these laws by means of the properties of right-angled triangles as set forth in Art. 26; and, accordingly, this method is adopted here. These laws are also derived directly from geometry in Note A at the end of the book. It may be stated here that the geometrical derivation will strengthen the student's understanding of the subject, and will show more clearly the correspondence (Art. 14) between the parts of a spherical triangle and the parts of a triedral angle.

36. The *Law of Sines* and the *Law of Cosines* deduced by means of the relations of right-angled triangles.

***A*. Derivation of the Law of Sines.**

Let ABC (Figs. 39, 40) be any spherical triangle. From B draw the arc BD at right angles to AC to meet AC, or AC produced, in D.

In ABD, $\sin BD = \sin c \sin A$;

in CBD, $\sin BD = \sin a \sin C$ (Fig. 39)

$= \sin a \sin BCD$ (Fig. 40) $= \sin a \sin BCA$.

Hence, in both figures, $\sin a \sin C = \sin c \sin A$.

$$\therefore \frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}.$$

Similarly, by drawing an arc from C at right angles to AB, it can be shown that $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}$.

$$\therefore \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \quad (1)$$

In words: *In a spherical triangle the sines of the sides are proportional to the sines of the opposite angles.* (Compare *Plane Trigonometry*, Art. 54, I.)

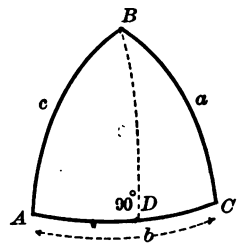

FIG. 39

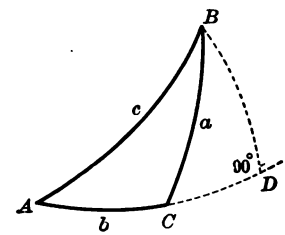

FIG. 40

***B*. Derivation of the Law of Cosines.**

$$\begin{aligned}\cos BC &= \cos CD \cos DB \\ &= \cos(b - AD)\cos DB, \text{ or } \cos(AD - b)\cos DB \\ &= \cos b \cos AD \cos DB + \sin b \sin AD \cos DB. \quad (a)\end{aligned}$$

But $$\cos AD \cos DB = \cos c;$$

and $$\sin AD \cos DB = \frac{\cos c \sin AD}{\cos AD} = \cos c \tan AD$$

$$= \cos c \tan c \cos A = \sin c \cos A.$$

Hence, on substituting in (a),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \quad (2)$$

Similarly, or by taking the sides in turn,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

In words: *In a spherical triangle the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of their included angle.* (Compare *Plane Trigonometry*, Art. 54, II.)

NOTE 1. **The law of cosines, (2), is the fundamental and the most important relation in spherical trigonometry.** For, as shown in Note [illegible], it can be deduced directly; the law of sines, (1), can be deduced from it; all other relations follow from these; and the relations for right triangles, Art. 26, can be deduced from the relations for triangles in general, on letting C be a right angle. The formulas for $\cos a$, $\cos b$, $\cos c$, were known to the Arabian astronomer Al Battani in the ninth century. (See *Plane Trigonometry*, p. 166.)

***C*. The Law of Cosines for the angles.** Relation (2) holds for all triangles, and, accordingly, for $A'B'C'$, the polar triangle of ABC. (See Fig. 14, Art. 16.) That is,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

$$\therefore \cos (180° - A) = \cos (180° - B) \cos (180° - C)$$

$$+ \sin (180° - B) \sin (180° - C) \cos (180° - a). \quad [\text{Art. 16. } d.]$$

$$\therefore -\cos A = (-\cos B)(-\cos C) + \sin B \sin C (-\cos a).$$

$$\therefore \boldsymbol{\cos A = -\cos B \cos C + \sin B \sin C \cos a;} \qquad (3)$$

Similarly, $\cos B = -\cos C \cos A + \sin C \sin A \cos b$,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Relation (3) can also be derived by means of right-angled triangles.

NOTE 2. From (2), $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$.

The denominator in the second member is always positive. *If a differs more from 90° than does b*, then $\cos a$ is numerically greater than $\cos b$, and, accordingly, greater than $\cos b \cos c$; hence $\cos A$ and $\cos a$ have the same sign, and thus, *a and A are in the same quadrant.*

Similarly, a and A are in the same quadrant when a differs more from 90° than does c.

From (3), in which $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$,

it can be shown in a similar way that *if A differs more from 90° than does B or C, then a and A are in the same quadrant.*

Ex. 1. Derive $\cos b$ and $\cos c$ by means of right triangles.

Ex. 2. Derive $\cos A$ and $\cos B$ by means of right triangles.

Ex. 3. Derive $\cos C$ from $\cos c$ by means of the polar triangle.

37. Formulas for the half-angles and the half-sides.

[Compare the method and results of this article with those of Art. 62, *Plane Trigonometry*].

I. The half-angles.

From Art. 36, (2), $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$. (1)

$$\therefore\ 1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

$$\therefore\ 2 \sin^2 \tfrac{1}{2} A = \frac{2 \sin \tfrac{1}{2}(a - b + c) \sin \tfrac{1}{2}(a + b - c)}{\sin b \sin c}.$$

[*Plane Trigonometry*, Art. 52, (8).]

On putting $a + b + c = 2s$, then $-a + b + c = 2(s - a)$,

$a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$.

$$\therefore\ \sin^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}. \qquad (2)$$

Similarly, from (1),

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a + \sin b \sin c - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

$$\therefore\ 2 \cos^2 \tfrac{1}{2} A = \frac{2 \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(b + c - a)}{\sin b \sin c}.$$

$$\therefore\ \cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s - a)}{\sin b \sin c}. \qquad (3)$$

$$\left.\begin{aligned} \sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}; \\ \cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}; \\ \text{and hence,}\quad \tan\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}. \end{aligned}\right\} \quad (4)$$

Therefore, $$\tan\tfrac{1}{2}A = \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

$$\left.\begin{aligned} \text{hence, if}\quad \tan r &= \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}, \\ \text{then}\quad \tan\tfrac{1}{2}A &= \frac{\tan r}{\sin(s-a)}. \end{aligned}\right\} \quad (5)$$

Like formulas can be similarly derived for $\frac{1}{2}B$ and $\frac{1}{2}C$; or they may be written immediately on observing the symmetry in formulas (4) and (5); namely,

$$\left.\begin{aligned} \sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}}, & \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}, \\ \cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}}, & \cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}, \\ \tan\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}}, & \tan\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}. \end{aligned}\right\} \quad (6)$$

$$\tan\tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)}, \qquad \tan\tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)}. \quad (7)$$

It is shown in Art. 50 that r is the radius of the circle inscribed in the triangle ABC. Article 50 may be read at this time.

Note. By geometry, $2s < 360°$ and $b + c > a$. Hence, $s - a$ is positive and less than 180°. Similarly, $s - b$, $s - c$, are positive. Therefore, the quantities under the radical signs are positive. The positive signs must be given to each radical, for A, B, C, are each less than 180°, and, consequently, $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, are each between 0° and 90°.

EXAMPLES.

1. Derive each of the above formulas.

2. Given $a = 58°$, $b = 80°$, $c = 96°$. Find A, B, C.

3. Given $a = 46° 30'$, $b = 62° 40'$, $c = 83° 20'$. Find A, B, C.

The results in Exs. 2, 3, may be *checked* by Art. 36, (1).

II. The half-sides. From Art. 36, (3),

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

On finding $1 - \cos a$ and $1 + \cos a$, combining and simplifying in the manner followed for the half-angles, and putting

$$A + B + C = 2S,$$

the following formulas are obtained:

$$\left.\begin{aligned} \sin\tfrac{1}{2}a &= \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}; \\ \cos\tfrac{1}{2}a &= \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}; \\ \tan\tfrac{1}{2}a &= \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}. \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} \text{Let} \quad \tan R &= \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}, \\ \text{then} \quad \tan\tfrac{1}{2}a &= \tan R \cos(S-A). \end{aligned}\right\} \quad (9)$$

Similarly, or from (8) and (9) by symmetry,

$$\left.\begin{aligned} \sin\tfrac{1}{2}b &= \sqrt{\frac{-\cos S \cos(S-B)}{\sin A \sin C}}, & \sin\tfrac{1}{2}c &= \sqrt{\frac{-\cos S \cos(S-C)}{\sin A \sin B}}, \\ \cos\tfrac{1}{2}b &= \sqrt{\frac{\cos(S-A)\cos(S-C)}{\sin A \sin C}}, & \cos\tfrac{1}{2}c &= \sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A \sin B}}, \\ \tan\tfrac{1}{2}b &= \sqrt{\frac{-\cos S \cos(S-B)}{\cos(S-A)\cos(S-C)}}, & \tan\tfrac{1}{2}c &= \sqrt{\frac{-\cos S \cos(S-C)}{\cos(S-A)\cos(S-B)}}. \end{aligned}\right\} \quad (10)$$

$$\tan\tfrac{1}{2}b = \tan R \cos(S-B), \qquad \tan\tfrac{1}{2}c = \tan R \cos(S-C). \quad (11)$$

It is shown in Art. 49 that R is the radius of the circumscribing circle of the triangle ABC. Article 49 may be read at this time.

NOTE 1. Formulas (8)–(11) can also be derived from formulas (4)–(7) by making use of the polar triangle, as done in Art. 36. *C.*

NOTE 2. Since $A + B + C$ lies between 180° and 540° (Art. 17), S lies between 90° and 270°; hence, $\cos S$ is negative, and, accordingly, $-\cos S$ is positive. Since all the other functions under the radical signs are positive, the whole expression under each radical sign is positive.

NOTE 3. The positive value of the radical is taken, since each side (Art. 19) is less than 180°

EXAMPLES.

1. Derive formulas (10) from the values of $\cos b$ and $\cos c$.
2. Derive formulas (10) from formulas (6) by means of the polar triangle.
3. In ABC, given $A = 78° 40'$, $B = 63° 50'$, $C = 46° 20'$. Find a, b, c.

[SUGGESTION. Either use formulas (8)–(10); or, solve the polar triangle, and thence obtain the parts of the original triangle. [*The results may be checked by using both these methods*, or by Art. 36, (1).]

4. In ABC, given $A = 121°$, $B = 102°$, $C = 68°$. Find a, b, c.
5. Show that $\cos(S - A)$ is positive.

38. Napier's Analogies. On *dividing* $\tan\frac{1}{2}A$ by $\tan\frac{1}{2}B$ (Art. 37), there is obtained,

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B} = \frac{\sin(s-b)}{\sin(s-a)}.$$

From this, by composition and division,

$$\frac{\tan\frac{1}{2}A + \tan\frac{1}{2}B}{\tan\frac{1}{2}A - \tan\frac{1}{2}B} = \frac{\sin(s-b) + \sin(s-a)}{\sin(s-b) - \sin(s-a)}.$$

This, by *Plane Trigonometry*, Arts. 44. *B*, 52 (also, see Art. 61), reduces to

$$\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B + \cos\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}A\cos\frac{1}{2}B - \cos\frac{1}{2}A\sin\frac{1}{2}B} = \frac{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{2\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)};$$

and thence, to

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}. \qquad (1)$$

On *multiplying* $\tan\frac{1}{2}A$ by $\tan\frac{1}{2}B$, there is obtained

$$\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B = \frac{\sin(s-c)}{\sin s},$$

i.e.

$$\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B} = \frac{\sin(s-c)}{\sin s}.$$

From this, by composition and division,

$$\frac{\cos\frac{1}{2}A\cos\frac{1}{2}B - \sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}B + \sin\frac{1}{2}A\sin\frac{1}{2}B} = \frac{\sin s - \sin(s-c)}{\sin s + \sin(s-c)} = \frac{2\cos\frac{1}{2}(2s-c)\sin\frac{1}{2}c}{2\sin\frac{1}{2}(2s-c)\cos\frac{1}{2}c}.$$

Whence,

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}. \qquad (2)$$

Either, on proceeding in a similar way with $\tan \frac{1}{2}a$ and $\tan \frac{1}{2}b$ [Art. 37, (8), (10)], *or*, on applying (1) and (2) to the polar triangle, there is obtained,

$$\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)} \tag{3}$$

and

$$\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A+B)}. \tag{4}$$

Relations (1)–(4) are known as *Napier's Analogies.**

NOTE 1. Compare (3) with formula (2) Art. 61, *Plane Trigonometry.*

NOTE 2. The numerators in (3) are always positive, since $a+b+c<360°$ and $C<180°$. It follows, accordingly, that $a-b$ *and* $A-B$ *must have the same sign.* This also follows from the geometrical fact [Art. 15, (5)] that the greater angle is opposite the greater side.

NOTE 3. In relation (4), $\cot \frac{1}{2}C$ and $\cos \frac{1}{2}(a-b)$ are positive quantities; hence $\cos \frac{1}{2}(a+b)$ and $\tan \frac{1}{2}(A+B)$ have the same sign; and, accordingly, $\frac{1}{2}(a+b)$ *and* $\frac{1}{2}(A+B)$ *are of the same species* (Art. 27).

NOTE 4. *Derivation of* (3) *by applying* (1) *to the polar triangle.* On applying (1) to the polar triangle $A'B'C'$ (Fig. 14, Art. 16),

$$\frac{\sin \frac{1}{2}(A'+B')}{\sin \frac{1}{2}(A'-B')} = \frac{\tan \frac{1}{2}c'}{\tan \frac{1}{2}(a'-b')}.$$

$$\therefore \frac{\sin \frac{1}{2}(\overline{180°-a}+\overline{180°-b})}{\sin \frac{1}{2}(\overline{180°-a}-\overline{180°-b})} = \frac{\tan \frac{1}{2}(180°-C)}{\tan \frac{1}{2}(\overline{180°-A}-\overline{180°-B})}; \quad \text{[Art. 16. } d.\text{]}$$

i.e.

$$\frac{\sin(180°-\frac{1}{2}\overline{a+b})}{\sin \frac{1}{2}(b-a)} = \frac{\tan(90°-\frac{1}{2}C)}{\tan \frac{1}{2}(B-A)}.$$

Whence,

$$\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)}.$$

NOTE 5. For a *geometrical deduction* of Napier's Analogies and the formulas in Art. 39, see M'Clelland and Preston, *Treatise on Spherical Trigonometry*, Part I., Art. 56, and the article *Trigonometry* in the Encyclopedia Britannica (9th edition).

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* That is, Napier's *proportions.* For a long time the word *analogy* was used in English in one of its original Greek meanings, namely, a proportion (*i.e.* an equality of ratios). This use of the word is now obsolete, and is only retained in a few phrases such as the above. Napier (see Art. 30, and *Plane Trigonometry*, Art. 1) discovered these proportions and gave them in his work, *Mirifici logarithmorum canonis descriptio*, in 1614

EXAMPLES.

1. Express Napier's Analogies in *words*.
2. *Write* the analogies involving B and C, A and C, b and c, a and c.
3. *Derive* some of the analogies in Ex. 2.

39. Delambre's Analogies or Gauss's Formulas.

By *Plane Trigonometry*, Art. 46, (1),

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B.$$

By Art. 37, (4), (6),

$$\sin \tfrac{1}{2}A \cos \tfrac{1}{2}B = \frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} = \frac{\sin(s-b)}{\sin c}\cos \tfrac{1}{2}C,$$

and

$$\cos \tfrac{1}{2}A \sin \tfrac{1}{2}B = \frac{\sin(s-a)}{\sin c}\sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} = \frac{\sin(s-a)}{\sin c}\cos \tfrac{1}{2}C.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\sin(s-a)+\sin(s-b)}{\sin c}\cos \tfrac{1}{2}C$$

$$= \frac{2\sin \tfrac{1}{2}(2s-a-b)\cos \tfrac{1}{2}(a-b)}{2\sin \tfrac{1}{2}c \cos \tfrac{1}{2}c}\cos \tfrac{1}{2}C.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}c}\cos \tfrac{1}{2}C. \qquad (1)$$

In a similar way it may be shown that

$$\sin \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}c}\cos \tfrac{1}{2}C, \qquad (2)$$

$$\cos \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a+b)}{\cos \tfrac{1}{2}c}\sin \tfrac{1}{2}C, \qquad (3)$$

$$\cos \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a+b)}{\sin \tfrac{1}{2}c}\sin \tfrac{1}{2}C. \qquad (4)$$

Formulas (1)–(4) are known as Delambre's Analogies, and also as Gauss's Formulas or Equations.*

* These formulas were discovered by Karl Friedrich Gauss (1777–1855), one of the greatest of German mathematicians and astronomers, and published without proof in his *Theoria Motus Corporum Cœlestium* in 1809; thus they bear his name. They were, however, published earlier by Karl Brandon Mollweide of Leipzig (1774–1825) in Zach's *Monatliche Correspondenz* for November, 1808. They were earliest discovered by Jean Baptiste Joseph Delambre (1749–1822), a great French astronomer, in 1807, and published in the *Connaissance des Temps* in 1808. The geometrical proof (see Note 5, Art. 38) was the one originally given by Delambre. This proof was rediscovered and announced by M. W. Crofton in 1869, and published in the *Proceedings of the London Math. Soc.*, Vol. III. (1869–1871), p. 13.

NOTE 1. Equations (3) and (4) can also be derived by applying (1) and (2) to the polar triangle.

NOTE 2. Delambre's Analogies can also be deduced by help of Napier's Analogies. (See Todhunter, *Spherical Trigonometry*, Art. 54; *Nature*, Vol. XL. (1889, Oct. 31), p. 644.)

NOTE 3. On the other hand, Napier's Analogies can be easily derived from Delambre's Analogies; namely, on dividing corresponding members, one by the other, in (1) and (3), (2) and (4), (4) and (3), (2) and (1).

EXAMPLES.

1. Write Delambre's Analogies involving B and C, and C and A.
2. Derive (3) and (4) from (1) and (2), using the polar triangle.
3. Derive Delambre's Analogies from Napier's Analogies.
4. Derive some of the analogies in Ex. 1 directly.

40. Other relations between the parts of a spherical triangle. The preceding articles of this chapter present few more relations than are required for the solution of spherical triangles. Between the parts of a spherical triangle there are many other relations which are interesting and useful for many purposes, and which either set forth, or lead to the discovery of, important geometrical properties * of spherical triangles.

For example, if in equation (2) Art. 36, the value of $\cos c$ in the second equation that follows, be substituted, then

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A;$$

whence, on putting for $\cos^2 b$ its value $1 - \sin^2 b$, dividing by $\sin b$, and transposing, it follows that

$$\cos a \sin b - \sin a \cos b \cos C = \sin c \cos A.$$

Five similar relations can be derived by permuting the letters; and on applying these six relations to the polar triangle, six others can be derived.

To pursue this topic further is beyond the scope of this book, which aims to give little more than the simplest elements of spherical trigonometry and what is absolutely required for the solution of spherical triangles. Those who are interested can refer to the works on spherical trigonometry by M'Clelland and Preston (Macmillan & Co.), Casey (Longmans, Green, & Co.), Bowser (D. C. Heath & Co.), and others.

N.B. *Questions and exercises on Chapter III. will be found on page* 104.

* Instances in which geometrical properties are deduced by means of trigonometry, are given in Art. 27, Art. 36, (Note 2), Art. 38, (Notes 2, 3).

CHAPTER IV.

SOLUTION OF TRIANGLES.

N.B. The student is recommended to work one or two examples in each set in this chapter *before* reading any of the text.

41. Cases for solution. This chapter is concerned with the numerical solution of spherical triangles. In all there are six cases for solution; these correspond respectively to the six cases for construction which were discussed in Arts. 23, 24. In these cases the given parts are as follows:

I. **Three sides.**

II. **Three angles.**

III. **Two sides** and **their included angle.**

IV. **One side** and **its two adjacent angles.**

V. **Two sides** and **the angle opposite one of them.**

VI. **Two angles** and **the side opposite one of them.**

With slight changes the procedure described in Art. 31 is recommended. Figures may be helpful. Of formulas adapted for logarithmic computation, the necessary ones are (1) Art. 36, (4)–(11) Art. 37, and (1)–(4) Art. 38. If the polar triangle is used in finding the solution, then I. and II. constitute one case, likewise III. and IV., and likewise V. and VI.; and the necessary formulas are (1) Art. 36 (4)–(7) *or* (8)–(11) Art. 37, and (1), (2), *or* (3), (4) Art. 38. Cases V. and VI. must be examined as to ambiguity; and accordingly, they give more trouble than the others. *Unless the triangle satisfies the conditions specified in Arts* 15, 17, 24 V., *its solution is impossible.*

Checks. **The results obtained should always be checked.** Delambre's Analogies and formulas which have not been used in the course of the solution, may be used as check formulas.

N.B. Before doing any of the numerical work the student should try to get a clear idea of the figure of the triangle upon a sphere. This geometrical

conception will enable him to make a reasonable estimate of what the results will be; this estimate will help him to detect wild results that may be obtained by making numerical errors. For example, in ABC let $a = 110°$, $b = 114°$, $C = 10°$; and suppose that the result $c = 76°$ presents itself. A person who has drawn a figure of the triangle on a sphere, or one who has geometrical imagination sufficient to give him an idea of the look of the given triangle, will at once see that the result, $c = 76°$, must be wrong. In working spherical triangles it is much better not to proceed blindly.

42. Case I. Given the three sides.

EXAMPLES.

1. In ABC, given $a = 47° 30'$, $b = 55° 40'$, $c = 60° 10'$. Find A, B, C.

Formulas: $$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \tfrac{1}{2}A = \frac{\tan r}{\sin(s-a)},\quad \tan \tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)},\quad \tan \tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)}.$$

Check: Law of Sines, or Napier's Analogies, or Delambre's Analogies.

Logarithmic formulas:

$$\log \tan^2 r = \log\sin(s-a) + \log\sin(s-b) + \log\sin(s-c) - \log\sin s, \text{ etc.}$$

Check: $\log\sin a - \log\sin A = \log\sin b - \log\sin B = \log\sin c - \log\sin C$.

$a = 47° 30'$	$\log\sin s = 9.99539 - 10$	$\therefore \frac{1}{2}A = 28° 16' 2''$
$b = 55° 40'$	$\log\sin(s-a) = 9.74943 - 10$	$\frac{1}{2}B = 34° 33' 41.5''$
$c = 60° 10'$	$\log\sin(s-b) = 9.64184 - 10$	$\frac{1}{2}C = 39° 29' 12''$
$\therefore 2s = 163° 20'$	$\log\sin(s-c) = 9.56408 - 10$	$\therefore A = 56° 32' 4''$
$s = 81° 40'$	$\therefore \log\tan^2 r = 18.95996 - 20$	$B = 69° 7' 23''$
$s - a = 34° 10'$	$\therefore \log\tan r = 9.47998 - 10$	$C = 78° 58' 24''$
$s - b = 26°$	$\therefore \log\tan \frac{1}{2}A = 9.73055 - 10$	
$s - c = 21° 30'$	$\log\tan \frac{1}{2}B = 9.83814 - 10$	
	$\log\tan \frac{1}{2}C = 9.91590 - 10$	

Check:	$\log\sin a = 9.86763$	$\log\sin b = 9.91686$	$\log\sin c = 9.93826$
	$\log\sin A = \underline{9.92128}$	$\log\sin B = \underline{9.97051}$	$\log\sin C = \underline{9.99191}$
	9.94635	9.94635	9.94635

NOTE 1. *Directions for the numerical work:* Fill in the first column; turn up the first four logarithms in the second column (since these logarithms are required by the formulas); compute the last five logarithms in the second column according to the formulas (these computations may be made on another paper, if necessary); find the first three angles of the third column by the tables; thence compute A, B, C.

NOTE 2. If only one angle is required, say A, it can be found by one of formulas (4) Art. 37; preferably, the second. Angle A can also be found (without logarithms) by formula (1) Art. 37.

2. Solve ABC, given that $a = 43° 30'$, $b = 72° 24'$, $c = 87° 50'$.

3. Solve ABC, given that $a = 110° 40'$, $b = 45° 10'$, $c = 73° 30'$.

4. Solve ABC, given that $a = 120° 50'$, $b = 98° 40'$, $c = 74° 60'$.

5. Solve PQR, given that $p = 67° 40'$, $q = 47° 20'$, $r = 83° 50'$.

43. Case II. Given the three angles.

Either: Solve the polar triangle by the method used in Case I, and therefrom obtain the parts of the original triangle.

Or: Solve by means of formulas (8)–(11) Art. 37.

EXAMPLES.

Solve ABC, and check the results.

1. Given $A = 74° 40'$, $B = 67° 30'$, $C = 49° 50$.

2. Given $A = 112° 30'$, $B = 83° 40'$, $C = 70° 10'$.

3. Given $A = 130°$, $B = 98°$, $C = 64°$.

4. Given $P = 33° 40'$, $Q = 26° 10'$, $R = 20° 30'$. Find p, q, r.

NOTE. The results may also be checked by solving the examples by both the methods above.

44. Case III. Given two sides and their included angle.

EXAMPLES.

1. In ABC, $a = 64° 24'$, $b = 42° 30'$, $C = 58° 40'$; find A, B, c.

Formulas

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\sin c = \frac{\sin a}{\sin A} \sin C.$$

Checks: Law of Sines, etc.

$C = 58° 40'$ $\quad \log \cot \tfrac{1}{2} C = 0.25031 \quad \therefore \log \tan \tfrac{1}{2}(A+B) = 0.46743$

$a = 64° 24'$ $\quad \log \sin \tfrac{1}{2}(a+b) = 9.90490 - 10 \quad \log \tan \tfrac{1}{2}(A-B) = 9.62405$

$b = 42° 30'$ $\quad \log \cos \tfrac{1}{2}(a+b) = 9.77490 - 10 \quad \therefore \tfrac{1}{2}(A+B) = 71° 10' 41'$

$\therefore a+b = 106° 54'$ $\quad \log \sin \tfrac{1}{2}(a-b) = 9.27864 - 10 \quad \tfrac{1}{2}(A-B) = 22° 49' 12'$

$a-b = 21° 54'$ $\quad \log \cos \tfrac{1}{2}(a-b) = 9.99202 - 10 \quad \therefore A = 93° 59' 53'$

$\tfrac{1}{2} C = 29° 20'$ $\quad \log a = 9.95513 - 10 \quad B = 48° 21' 29'$

$\tfrac{1}{2}(a+b) = 53° 27'$ $\quad = 9.93154 - 10 \quad \therefore \log \sin A = 9.99894 -$

$\tfrac{1}{2}(a-b) = 10° 57'$ $\quad \therefore \log \sin c = 9.88773 -$

$\therefore c = 50° 33' 6''$

NOTE 1. Since $C<A$, then $c<a$; and hence, the acute value of c is taken.

NOTE 2. *Directions for the numerical work:* Fill in the first column; then turn up *all* the logarithms for the second column, these logarithms being required by the formulas; then compute the first two logarithms in the third column, according to the formulas; thence find the corresponding angles, and calculate A and B; turn up $\log \sin A$; compute $\log \sin c$ according to the formula; then find c in the Tables.

NOTE 3. In using formulas involving the difference of two sides or two angles, place the larger side or angle first.

2. Solve ABC, given $a = 93° 20'$, $b = 56° 30'$, $C = 74° 40'$.

3. Solve ABC, given $b = 76° 30'$, $c = 47° 20'$, $A = 92° 30'$.

4. Solve ABC, given $c = 40° 20'$, $a = 100° 30'$, $B = 46° 40'$.

5. Solve PQR, given $q = 76° 30'$, $r = 110° 20'$, $P = 46° 50'$.

45. Case IV. Given one side and its two adjacent angles.

Either: Solve the polar triangle by the method used in Case III.; and therefrom obtain the parts of the original triangle.

Or: Solve by using formulas (1), (2), Art. 38.

EXAMPLES.

1. Solve ABC, given $A = 67° 30'$, $B = 45° 50'$, $c = 74° 20'$.

2. Solve ABC, given $B = 98° 30'$, $C = 67° 20'$, $a = 60° 40'$.

3. Solve ABC, given $C = 110°$, $A = 94°$, $b = 44°$.

4. Solve PQR, given $R = 70° 20'$, $Q = 43° 50'$, $p = 50° 46'$.

46. Case V. Given two sides and the angle opposite one of them.

This is an **ambiguous case,*** since (Art. 24, V.) there may be two solutions. It may be well to examine this case (1) *geometrically*, that is, by an inspection of the figure; (2) *analytically*, that is, by an inspection of the formulas involved in its solution.

(1) *Geometrically.* In Art. 24, V. (Figs. 21, 25) it has been seen that, when two sides and an angle opposite one of them (say, a, b, A) of a triangle ABC are given, there are *two* triangles possible if either of the following sets of conditions holds, viz.:

$$A < 90°,\ a > p,\ a < b, \text{ and } a < 180° - b; \qquad (a)$$

$$A > 90°,\ a < p,\ a > b, \text{ and } a > 180° - b. \qquad (b)$$

* For a detailed discussion of the ambiguous case, see Todhunter, *Spherical Trigonometry*, pp. 53–58; M'Clelland and Preston, *Spherical Trigonometry*, pp. 137–143.

In order that the triangle be *possible*, it is apparent that: *either* $CB = CP$; or, in Fig. 21, $CB > CP$, *i.e.* $\sin CB > \sin CP$, *i.e.* $\sin a > \sin AC \sin CAP$,

$$i.e.\ \sin a > \sin b \sin A;$$

and, in Fig. 25, $CLB < CP$, and $CLB > CP'$, *i.e.* $\sin a > CP'$, *i.e* $\sin a > \sin AC \sin CAP'$,

$$i.e.\ \sin a > \sin b \sin (180° - CAP),\ i.e.\ \sin a > \sin b \sin A.$$

Art. 24 also shows that, when the triangle is possible, there is *one* solution if either of the following sets of conditions holds, viz.:

$A < 90°$, $a > p$, a between b and $180° - b$; (*c*)

$A > 90°$, $a < p$, a between b and $180° - b$. (*d*)

If $CB = CP$, *i.e.* if $a = p$, then there is *one* solution.

Art. 24 also shows that the triangle is impossible if either one of the following sets of conditions holds, viz.:

$A < 90°$, a greater than both b and $180° - b$; (*e*)

$A > 90°$, a less than both b and $180° - b$.

Since the greater angle is opposite the greater side, B must be such that $A - B$ shall have the same sign as $a - b$.

(2) *Analytically.* The formulas used in solving this case are as follows:

$$\sin B = \frac{\sin b \sin A}{\sin a}, \tag{1}$$

$$\cot \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(a+b)}{\sin \tfrac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B), \quad [\text{or, (4) Art. 38}] \tag{2}$$

$$\tan \tfrac{1}{2} c = \frac{\sin \tfrac{1}{2}(A+B)}{\sin \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b). \quad [\text{or, (2) Art. 38}] \tag{3}$$

Since B is determined from its sine, it may be in either the first or the second quadrant. If $\sin a = \sin b \sin A$, then $B = 90°$. If $\sin a < \sin b \sin A$, then $\sin B > 1$, and B has an impossible value, and, accordingly, the triangle is impossible. [Compare above.] Equation (2) shows that $A - B$ and $a - b$ have the same sign.

Hence, from the *analytical* inspection comes the following **rule:**

If $\sin a < \sin b \sin A$, *there is* **no** *solution; if* $\sin a = \sin b \sin A$, *there is* **one** *solution; if* $\sin a > \sin b \sin A$, *and if* **both** *values of* B *obtained from* (1) *be such that*

$$A - B \text{ and } a - b \text{ have like signs,}$$

there are **two** *solutions; if only* **one** *of the values of* B *satisfies this condition, there is only* **one** *solution; if* **neither** *of the values of* B *satisfies this condition, the solution is* **impossible.**

From the *geometrical* inspection comes the following **rule:**

If $\sin a < \sin b \sin A$, *there is* **no** *solution; if* $\sin a = \sin b \sin A$, *there is* **one** *solution; if* $\sin a > \sin b \sin A$, *then:*

When A is less than 90°:

there are **two** *solutions if* a *is* **less** *than both* b *and* $180° - b$;
there is **one** *solution if* a *lies* **between** b *and* $180° - b$;
there is **no** *solution if* a *is* **greater** *than both* b *and* $180° - b$.

When A is greater than 90°:

there are **two** *solutions if* a *is* **greater** *than both* b *and* $180° - b$;
there is **one** *solution if* a *lies* **between** b *and* $180° - b$;
there is **no** *solution if* a *is* **less** *than both* b *and* $180° - b$.

NOTE 1. The second rule has one advantage over the first, in that it enables one to say, merely on calculating $\sin B$, but without finding B, whether the triangle is ambiguous or not.

NOTE 2. The property observed in Art. 36, Note 2, is frequently used in investigating the ambiguous case.

EXAMPLES.

1. In ABC, $a = 43° 20'$, $b = 48° 30'$, $A = 58° 40'$; find B, C, c.

Formulas: $$\sin B = \frac{\sin b \sin A}{\sin a}.$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)} \tan \tfrac{1}{2}(B - A). \qquad \text{[Art. 38, (3)]}$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a). \qquad \text{[Art. 38, (1)]}$$

Checks: Formulas (2), (4), Art. 38; or, formulas, Art. 37; or, Delambre's Analogies.

$A = 58° 40'$	$\log \sin A = 9.93154 - 10$	$\therefore B + A = 127° 27'$
$a = 43° 20'$	$\log \sin a = 9.83648 - 10$	$B - A = 10° 7'$
$b = 48° 30'$	$\log \sin b = 9.87446 - 10$	$\frac{1}{2}(B + A) = 63° 43' 30''$
$\therefore b + a = 91° 50'$	$\therefore \log \sin B = 9.96952 - 10$	$\frac{1}{2}(B - A) = 5° 3' 30''$
$b - a = 5° 10'$	$\therefore B = 68° 47'$	$\therefore B' + A = 169° 53'$
$\frac{1}{2}(b + a) = 45° 55'$	$B' = 111° 13'$	$B' - A = 52° 33'$
$\frac{1}{2}(b - a) = 2° 35'$	[According to the test for ambiguity.]	$\frac{1}{2}(B' + A) = 84° 56' 30''$
		$\frac{1}{2}(B' - A) = 26° 16' 30''$

In ABC. (*See Fig.* 21, *Art.* 24.)	*In AB'C.*
$\log \sin \frac{1}{2}(b + a) = 9.85632 - 10$	As in ABC.
$\log \sin \frac{1}{2}(b - a) = 8.65391 - 10$	
$\log \tan \frac{1}{2}(b - a) = 8.65435 - 10$	
$\log \sin \frac{1}{2}(B + A) = 9.95264 - 10$	$\log \sin \frac{1}{2}(B' + A) = 9.99830 - 10$
$\log \sin \frac{1}{2}(B - A) = 8.94532 - 10$	$\log \sin \frac{1}{2}(B' - A) = 9.64609 - 10$
$\log \tan \frac{1}{2}(B - A) = 8.94702 - 10$	$\log \tan \frac{1}{2}(B' - A) = 9.69345 - 10$
$\therefore \log \cot \frac{1}{2} C = 0.14943$	$\therefore \log \cot \frac{1}{2} C = 0.89586$
$\log \tan \frac{1}{2} c = 9.66167 - 10$	$\log \tan \frac{1}{2} c = 9.00656 - 10$
$\therefore \frac{1}{2} C = 35° 19' 55''.4$, $\frac{1}{2} c = 24° 38' 53''$.	$\therefore \frac{1}{2} C = 7° 14' 36''$, $\frac{1}{2} c = 5° 47' 49''$.
$\therefore C = 70° 39' 51''$, $c = 49° 17' 46''$.	$\therefore C = 14° 29' 12''$, $c = 11° 35' 38''$.

Hence, the solutions are:

$$ABC = 68° 47',\ ACB = 70° 39' 51'',\ AB = 49° 17' 46'';$$
$$AB'C = 111° 13',\ ACB' = 14° 29' 12'',\ AB' = 11° 35' 38''.$$

Note 3. *Directions for the numerical work:* Fill in the first of the three columns; turn up the first three logarithms in the second column, these being required by the first formula; compute $\log \sin B$ according to the first formula; find B in the tables; *decide the question of ambiguity;* fill in the third column (only four lines when the triangle is not ambiguous). Turn up the first six logarithms in the first of the next two columns; compute the next two logarithms according to the formulas; find the corresponding values in the Tables; thence compute C and c. If the case is ambiguous, do the same work for the second triangle.

2. Solve ABC when $a = 56° 40'$, $b = 30° 50'$, $A = 103° 40'$.
3. Solve ABC when $a = 30° 20'$, $b = 46° 30'$, $A = 36° 40'$.
4. Solve ABC when $c = 74° 20'$, $a = 119° 40'$, $C = 88° 30'$.
5. Solve ABC when $b = 30° 10'$, $c = 44° 30'$, $B = 86° 50'$.
6. Solve PQR when $q = 42° 30'$, $r = 46° 50'$, $Q = 56° 30'$.

47. Case VI. Given two angles and the side opposite one of them. This is also an ambiguous case.

Either: Solve the polar triangle by the method used in Case V.; and therefrom obtain the parts of the original triangle.

Or: Solve by using formula (1) Art. 36, and Napier's Analogies.

The first rule (Art. 46) for determining ambiguity suits the case, if a, b, be substituted for A, B, therein. On making use of the polar triangle, it is found that the second rule can be adapted by substituting a, A, B, for A, a, b, respectively.

EXAMPLES.

1. Solve ABC when $A = 108° 40'$, $B = 134° 20'$, $a = 145° 36'$.
2. Solve ABC when $B = 36° 20'$, $C = 46° 30'$, $b = 42° 12'$.
3. Solve ABC when $C = 62° 10'$, $A = 23° 46'$, $c = 33° 50'$.
4. Solve STV when $T = 102° 50'$, $V = 81° 20'$, $t = 124° 30'$.

48. Subsidiary angles. Formulas can sometimes be adapted for logarithmic computation and the triangle solved, by the use of *subsidiary angles. For example*, in ABC let a, c, B be known, and b required. (See Fig. 35, Art. 33.)

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \qquad \text{(Art. 36, } B\text{)}$$

$$= \cos c (\cos a + \sin a \tan c \cos B).$$

On putting $\tan c \cos B = \tan \phi$, this becomes

$$\cos b = \cos c (\cos a + \sin a \tan \phi)$$

$$= \frac{\cos c (\cos a \cos \phi + \sin a \sin \phi)}{\cos \phi}$$

$$= \frac{\cos c \cos (a - \phi)}{\cos \phi}.$$

On referring to Fig. 35 it is seen that $BD = \phi$, that $DC = a - \phi$, and $\cos AD = \frac{\cos c}{\cos \phi}$; so that solving as above is equivalent to solving the triangle by dividing it into right-angled triangles.

N.B. *Questions and exercises on Chapter IV. will be found on page* 105.

CHAPTER V.

CIRCLES CONNECTED WITH SPHERICAL TRIANGLES.

49. The circumscribing circle. The circle passing through the vertices of a spherical triangle is called the *circumscribing circle*, or *circum-circle*, of the triangle. This circle can be constructed in somewhat the same manner as the circumscribing circle of a plane triangle.

Let ABC (Fig. 41) be a spherical triangle, and let R denote the radius (*i.e.* the polar distance, Art. 6) of its circumscribing circle. Bisect the arcs BC, CA, in L, M, respectively; and at L, M, draw arcs at right angles to BC, CA, respectively. The point O, at which these arcs meet, is the pole of the circumscribing circle.

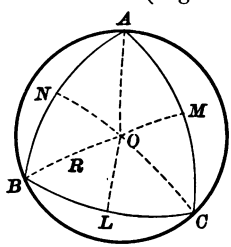

FIG. 41

For, draw OA, OB, OC, arcs of great circles. In the triangles OLB and OLC, $BL = LC$, LO is common, and the angles at L are right angles. Hence, $OB = OC$. In a similar way it can be shown that $OC = OA$. Hence O is the pole of the circumscribing circle.

Join O and N, the middle point of AB; then it is easily shown that ON is at right angles to AB.

In ABC, $\quad A + B + C = 2S.$

Now (since $OA = OB = OC$),

$$OAB = OBA, \quad OBC = OCB, \quad OCA = OAC.$$

Hence, $OAB + OBC + OAC = S.$

$$\therefore \quad OBC = S - (OAB + OAC) = S - A.$$

In the right-angled triangle OBL,

$$\tan OB = \frac{\tan BL}{\cos OBL}. \qquad \text{[Art. 26, Eq. (3)]}$$

$$i.e.\ \tan R = \frac{\tan \frac{1}{2} a}{\cos (S - A)}. \qquad (1)$$

Similarly, $\tan R = \dfrac{\tan \frac{1}{2} b}{\cos (S - B)}$, $\tan R = \dfrac{\tan \frac{1}{2} c}{\cos (S - C)}$.

On substituting in (1) the value of $\tan \frac{1}{2} a$ in relation (8) Art. 37, equation (1) becomes

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}. \qquad (2)$$

NOTE 1. Compare (1) with the corresponding case in plane triangles (*Plane Trig.*, Art. 68). (In plane triangles, $S = 90°$, and, hence, $\cos (S - A) = \sin A$.)

NOTE 2. On putting $N = \sqrt{-\cos S \cos(S - A)\cos(S - B)\cos(S - C)}$,

$$\tan R = -\frac{\cos S}{N}.$$

50. The inscribed circle. The circle which touches each of the sides of a spherical triangle is called the *inscribed circle*, or *in-circle*, of the triangle. This circle can be constructed in somewhat the same manner as the inscribed circle of a plane triangle.

Let ABC be a spherical triangle, and let r denote the radius (*i.e.* the polar distance) of its inscribed circle. Bisect angles A, B, by arcs of great circles, and let these arcs meet at O. Draw OL, OM, ON, at right angles to BC, CA, AB, respectively.

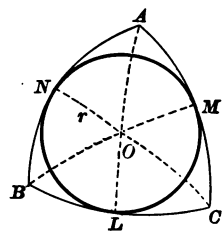

FIG. 42

In the triangles OAM and OAN, the angles at A are equal, the angles at N and M are right angles, and the side OA is common. Hence these triangles are symmetrical, and $OM = ON$. Similarly it can be shown that $ON = OL$. Hence O is the pole of the circle inscribed in ABC.

Since the triangles OAM and OAN are equal, $AM = AN$. Similarly, $BN = BL$, and $CL = CM$.

Now $$AB + BC + CA = 2s;$$

hence $$AN + BL + CL = s.$$

$$\therefore\ AN = s - (BL + LC) = s - a.$$

In the right-angled triangle AON,

$$\tan ON = \tan OAN \sin AN. \qquad \text{[Art. 26, (4)]}$$

$$\therefore \tan r = \tan \tfrac{1}{2} A \sin (s - a). \qquad (1)$$

Similarly, $\tan r = \tan \frac{1}{2} B \sin (s - b)$; $\tan r = \tan \frac{1}{2} C \sin (s - c)$.

On substituting in (1) the value of $\tan \frac{1}{2} A$ in (4) Art. 37, equation (1) becomes

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}. \qquad (2)$$

On putting $n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}$, (3)

$$\tan r = \frac{n}{\sin s}. \qquad (4)$$

NOTE 1. Compare (1) with *Plane Trigonometry*, Art. 69, Note; (2) with Art. 69, (3); n with S, Art. 66, (3); (4) with (3) Art. 69.

51. Escribed circles. A circle which touches a side of a spherical triangle, and the other two sides produced (that is, which is inscribed in a *co-lunar* triangle), is an *escribed circle*, or an *ex-circle*, of the triangle. There are three ex-circles, one corresponding to each side of the triangle.

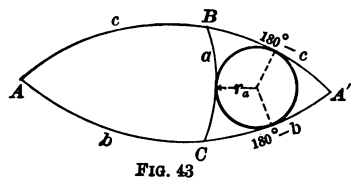

FIG. 43

Let ABC be a spherical triangle; and let the radii of the escribed circles, touching a, b, c, respectively, be denoted by r_a, r_b, r_c, respectively. Complete the lune whose angle is A. The escribed circle which touches a is the inscribed circle of the co-lunar triangle $A'BC$. Hence [Art. 50, (1)],

$$\tan r_a = \tan \tfrac{1}{2} A' \sin \tfrac{1}{2} [(a + \overline{180^\circ - b} + \overline{180^\circ - c}) - 2a];$$

i.e.

$$\tan r_a = \tan \tfrac{1}{2} A \sin s. \qquad (1)$$

Similarly, $\tan r_b = \tan \frac{1}{2} B \sin s$; $\tan r_c = \tan \frac{1}{2} C \sin s$.

On substituting for $\tan \frac{1}{2}A$ its value in (4) Art. 37, equation (1) becomes

$$\tan r_a = \sqrt{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}}, \qquad (2)$$

i.e. $$\tan r_a = \frac{n}{\sin (s-a)}. \qquad \text{[Art. 50, (3)]} \qquad (3)$$

Similarly, $\tan r_b = \frac{n}{\sin (s-b)}$; $\tan r_c = \frac{n}{\sin (s-c)}$.

NOTE. Compare (3) with the corresponding result in *Plane Trigonometry*, Art. 70.

Some other relations between the sides and angles of a spherical triangle and the radii of the circles connected with it, are indicated in the exercises at the end of the book.

EX. Find the radii of the circumscribing, inscribed, and escribed circles of some of the triangles in Chapters II., IV.

N.B. *For questions and exercises on Chapter V., see page* 107.

CHAPTER VI.

AREAS AND VOLUMES CONNECTED WITH SPHERES.

52. Preliminary propositions.

a. The lateral area of a frustum of a regular pyramid is equal to the product of the slant height of the frustum and half the sum of the perimeters of its bases.

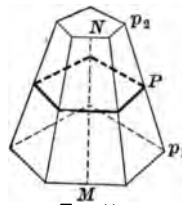

FIG. 44

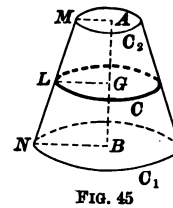

FIG. 45

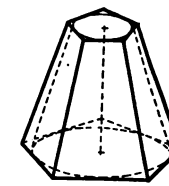
FIG. 46

The student can easily prove this (Fig. 44). It should be noted that the half sum of the perimeters of the bases of the frustum is equal to the perimeter of the section which is parallel to the bases and midway between them.

In symbols: If p_1, p_2, P, are the perimeters of the bases and the middle section of the frustum, and MN is its slant height, then

$$\text{lateral area of frustum} = \tfrac{1}{2}\, MN\,(p_1 + p_2) = MN \cdot P.$$

b. The lateral area of a frustum of a cone of revolution is equal to the product of the slant height of the frustum and half the sum of the circumferences of its bases.

[*Suggestion for proof:* If the number of the lateral faces of a frustum of a regular pyramid be indefinitely increased and each face be indefinitely decreased, then this frustum approaches the frustum of a cone of revolution as a limit (see Fig. 46). Accordingly, Proposition (*b*) follows at once from (*a*)]. It should be

noted that half the sum of the circumferences of the bases of the frustum is equal to the circumference of the section which is parallel to the bases and midway between them.

In symbols: If C_1, C_2, C (Fig. 45) are the circumferences of the bases and the middle section of the frustum, and MN is its slant height, then lateral area of frustum

$$= \tfrac{1}{2} MN\ (C_1 + C_2) = MN \cdot C = 2\ \pi LG \cdot MN.$$

NOTE. The lateral surface of the frustum of the cone (Fig. 45) can be generated by the revolution of the line MN about the line AB which is in the same plane with MN.

53. To find the area of a sphere. The surface of a sphere can be generated by the revolution of a semicircle about its diameter. For example, the semicircle $ATKB$ of radius R on revolving about its diameter AB, will describe the surface of a sphere of radius OA.

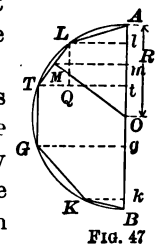

FIG. 47

Let a polygon $ALTGKB$ be inscribed in this semicircle. At M, the middle point of one of the chords LT, draw MO at right angles to LT. By geometry, MO will meet AB at O, the middle point of AB. Project LT on AB, the projection being lt; draw LQ at right angles to Tt.

By Art. 52.*b*, the area generated by LT in its revolution about AB

$$= 2\ \pi Mm \cdot LT. \qquad (1)$$

Since the angles of the triangle LTQ are respectively equal to the angles of OMm, these triangles are similar; accordingly,

$$LT : LQ = OM : Mm.$$

$$\therefore\ Mm \cdot LT = LQ \cdot OM = lt \cdot OM.$$

Hence, from (1), area generated by $LT = 2\ \pi OM \cdot lt.$ (2)

In words: When a chord of a semicircle revolves about the diameter, the area generated is equal to $2\ \pi$ times the product of the length of the perpendicular from the centre to the chord, and the projection of the chord upon the diameter.

$\therefore$ The area of the surface generated by the revolution of the polygon $ALTGKB$

$$= 2\pi \times (\text{perpendicular on } AL \text{ from } O) \times Al$$
$$+ 2\pi \times (\text{perpendicular on } LT \text{ from } O) \times lt$$
$$+ 2\pi \times (\text{perpendicular on } TG \text{ from } O) \times tg$$
$$+ 2\pi \times (\text{perpendicular on } GK \text{ from } O) \times gk$$
$$+ 2\pi \times (\text{perpendicular on } KB \text{ from } O) \times kB.$$

If the number of sides in the polygon inscribed in the semicircle is indefinitely increased and each side is indefinitely decreased, then the broken line $ALTGKB$ approaches the semicircle as a limit, and each of the perpendiculars drawn from O to the middle points of the chords approaches R as a limit; while the sum of the projections of the chords remains equal to AB, the diameter of the circle. Hence, area of surface generated by revolution of semicircle $AGB = 2\pi \cdot R \cdot 2R$;

i.e. area of surface of sphere of radius $R = 4\pi R^2$.

In words: The area of the surface of a sphere is four times the area of a great circle of the sphere.

Definition. A **zone** of a sphere is a portion of the surface included between two parallel planes, or, what comes to the same thing, is the portion of the surface included between two circles which have common poles; for example, the surface between the parallels of 30° N. latitude and 50° N. latitude.

The area of a zone. An infinite number of chords can be inscribed in the arc LT (Fig. 47). By reasoning similar to that employed above, it can be shown that

area of surface generated by arc $LT = 2\pi R \cdot lt$.

$\therefore$ *The area of a spherical zone is equal to the product of the length of a great circle of the sphere and the height of the zone.*

It follows that on a sphere or on equal spheres the areas of zones of equal heights are equal.

EXAMPLES.

1. Find the area of a sphere of radius 15 inches.

2. Find the surface of a spherical zone of height 2.5 inches on a sphere of diameter 50 inches.

3. Find the convex surface of a spherical segment of height 4.5 inches on a sphere of diameter 7 feet. [See definition, Art. 63.]

4. Suppose that the earth is a sphere whose radius is 3960 miles; find the area of the surface included between the North Pole and the parallel of 80° N. latitude; between the parallels of 49° N. and 50° N.; between 5° N. and 5° S.

54. Lunes. *Definition.* The spherical surface bounded by two halves of great-circles is called a *lune; e.g.* the surface between two meridians. The *angle of the lune* is the angle between the two semicircles; thus the angle of the lune between the meridians 70° W. and 80° W. is 10°.

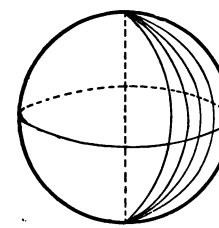

FIG. 48

Proposition. On the same circle or on equal circles the areas of lunes are proportional to their angles. This can be proved by a method similar to that which is used in proving that the angles at the centre of a circle are proportional to the arcs subtended by them.

55. A spherical degree defined. From the proposition in Art. 54 it follows that the area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles. That is,

area of lune of angle $A°$: area of sphere $= A° : 360°$.

Hence, $$\text{area of lune of angle } 1° = \frac{\text{area of sphere}}{360}.$$

Let a great circle be drawn about one of the vertices of a lune of angle 1° as a pole. The lune is then divided into two equal birectangular triangles; accordingly, each triangle contains $(\frac{1}{720})$th of the surface of the sphere, or $(\frac{1}{360})$th of the surface of the hemisphere. *The surface of each such triangle is called* **a spherical degree.**

For example, the part of the surface of a globe bounded by the meridians 43° W. and 63° W. longitude and the equator, contains 20 spherical degrees; the lune bounded by these meridians contains 40 spherical degrees.

A lune of angle $A°$ contains 2 A spherical degrees.

The passage from spherical degrees of surface to the ordinary measure (of the area) of the surface is easily effected when the radius of the sphere is given.

A spherical degree $= (\frac{1}{720})$*th part of the surface of a sphere;*

hence, **on a sphere of radius r,**

a spherical degree contains $\frac{4\pi r^2}{720}$, *i.e.* $\frac{\pi r^2}{180}$ **square units of area.**

Thus,

area of a lune of angle 20° on a sphere of radius $r = \frac{40\pi r^2}{180} = \frac{2}{9}\pi r^2$.

EXAMPLES.

1. Find the area of a lune of angle 10° on a sphere of radius 2 feet.

2. Find the area of a lune of angle 37° 30′ on a sphere of radius 7 feet.

3. Find the area between the meridians 77° W. and 83° 20′ W.; and the area between the meridians 174° 20′ W. and 158° 35′ E. (Radius of earth = 3960 miles.) [Express areas in spherical degrees and in square miles.]

56. Spherical excess of a triangle. The sum of the angles of a plane triangle is always *equal to* 180°; the sum of the angles of a spherical triangle is always *greater than* 180° (Art. 17). The difference between the latter sum and 180° is called *the spherical excess* of the triangle. (This *excess* is due to the fact that the triangle is spherical and not plane; hence the excess is called *spherical.*) For example, in the triangle bounded by the meridians 47° W. and 48° W. longitude and the equator, the sum of the angles is 181°; and, accordingly, the spherical excess is 1°. In the triangle bounded by the meridians 43° W. and 63° W. and the equator the sum of the angles is 200°, and the spherical excess is 20°; in the spherical triangle having angles 50°, 65°, 125°, the spherical excess is (50° + 65° + 125° − 180°), *i.e.* 60°.

If E denote the number of *degrees* in the spherical excess, and E_r denote the number of *radians* therein, then

in a triangle ABC, $E^\circ = A^\circ + B^\circ + C^\circ - 180^\circ$; (1)

and [*Plane Trigonometry*, Art. 73, (7)],

$$E_r = \left(\frac{A + B + C - 180}{180}\right)\pi. \quad (2)$$

Ex. Find the spherical excess (in *degrees* and in *radians*) of the triangles described in Art. 42, Exs. 1, 2, 3; Art. 43, Exs. 1, 2; Art. 44, Exs. 1, 2, 3; Art. 45, Exs. 1, 2; Art. 46, Exs. 1, 2, 3; Art. 47, Exs. 1, 2.

57. The area of a spherical triangle.

Proposition: The number of spherical degrees (of surface) *in a spherical triangle is equal to the number of* (angular) *degrees in its spherical excess.**

Let ABC be a spherical triangle whose spherical excess is E°; then area ABC is equal to E spherical degrees. Complete the great circle $BCB'C'$, and produce the arcs BA, CA to meet this circle in B', C', respectively. Complete the great circles $BAB'B$ and $ACA'C'$. The triangle $AB'C'$ is equal to the triangle $A'BC$. For,

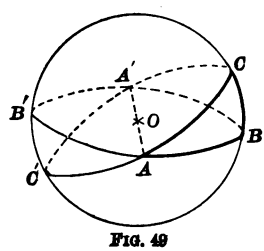

FIG. 49

$$B'A = 180^\circ - AB = BA',$$

$$C'A = 180^\circ - AC = CA',$$

$$C'B' = 180^\circ - B'C = CB.$$

Hence, *in area*, $ABC + AB'C' = \text{lune } ACA'BA$;

also $ABC + AB'C = \text{lune } BCB'AB$;

and $ABC + ABC' = \text{lune } CBC'AC$.

* This proposition is sometimes stated thus: *The area of a triangle is equal to its spherical excess*; but this enunciation is rather slipshod.

Hence, on addition,

$$2\,ABC + (ABC + AB'C' + AB'C + ABC')$$
$$= \text{lune } A + \text{lune } B + \text{lune } C;$$
$$\therefore \quad 2\,ABC = \text{lune } A + \text{lune } B + \text{lune } C - \text{hemisphere.}$$
$$\therefore \text{ (by Art. 55) } 2\,ABC = (2A + 2B + 2C - 360) \text{ spherical degrees.}$$
$$\therefore \quad ABC = (A + B + C - 180) \text{ spherical degrees}$$
$$= E \text{ spherical degrees.}$$

Since (Art. 55) a spherical degree on a sphere of radius r contains $\frac{1}{180}\pi r^2$ square units of area, then, on this sphere,

$$\text{area } ABC = \frac{A + B + C - 180}{180}\pi r^2 * = \frac{E}{180}\pi r^2, \qquad (1)$$
$$= E_r r^2, \qquad [\text{Art. 56 (2)}] \quad (2)$$

in which E denotes the number of degrees, and E_r denotes the number of radians in the spherical excess.

Hence, in order to find the area of a triangle, find the angles, calculate the spherical excess in degrees or radians, and use one of formulas (1), (2).

NOTE. It should be observed that [from Art. 14, Art. 56 (1), and the proposition above], *the number of spherical degrees* contained in the area subtended on a spherical surface by a solid angle at the centre of the sphere, *remains the same*, however the radius may vary. On the other hand, by (1) and (2), *the number of square units* in the subtended area *varies* as the square of the radius.

---

* This expression for the area of a spherical triangle was first given in 1629 by Albert Girard (1590–1634) (see *Plane Trigonometry*, pp. 22, 167); and it is often called *Girard's Theorem*. The method of proof used above was invented by John Wallis (1616–1703) professor of geometry at Oxford. (See Wallis, *Works*, Vol. II., p. 875.)

It follows from (1) that

$$\text{area } ABC : 2\pi R^2 = E^\circ : 360^\circ.$$

Hence, the above proposition may be expressed thus: *The area of a spherical triangle is to the surface of the hemisphere as the excess of its three angles above two right angles is to four right angles.*

EXAMPLES.

Find the areas of the following triangles (see examples, Art. 56):

1. Those described in Art. 42, Exs. 1, 2, 3, when on a sphere of radius 10 feet.

2. Those described in Art. 43, Exs. 1, 2, when on a sphere of radius 25 inches.

3. Those described in Art. 44, Exs. 1, 2, 3, when on a sphere of radius 30 yards.

4. Those described in Art. 45, Exs. 1, 2, when on a sphere of radius 4 feet.

5. Those described in Art. 46, Exs. 1, 2, 3, when on a sphere of radius 18 inches.

6. Those described in Art. 47, Exs. 1, 2, when on a sphere of radius 3960 miles.

58. Formulas for the spherical excess ($E°$) of a triangle. Since, in a spherical triangle ABC, $E° = A° + B° + C° - 180°$, and since there are many relations between the sides and angles of a triangle, it may be expected that there can be many formulas for the spherical excess; and, accordingly, for the area of a spherical triangle. [It will be remembered that there are several formulas for the area of a plane triangle (*Plane Trigonometry*, Art. 66).] Following are some of the most important of these (the deduction of some of them is given in Note B):

A. The spherical excess in terms of the three sides.

(a) *L'Huillier's* formula:

$$\tan \tfrac{1}{4} E° = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

(b) *Cagnoli's* formula: $\sin \tfrac{1}{2} E° = \dfrac{n}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}$,

in which $$n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

(c) *De Gua's* formula*: $\cot \tfrac{1}{2} E° = \dfrac{1 + \cos a + \cos b + \cos c}{2 n}$.†

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* *Simon L'Huillier* (1750–1810), a Swiss mathematician and philosopher; *Antoine Cagnoli* (1743–1816), an Italian astronomer; *L'abbé Jean Paul de Gua* (1712–1786), a French philosopher.

† For the deduction of this formula see Chauvenet, *Trigonometry*, p. 230, and Crawley, *Trigonometry*, p. 166.

B. The spherical excess in terms of two sides and their included angle.

$$(d)\quad \tan\tfrac{1}{2}E^\circ = \frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin C}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C};$$

$$(e)\quad \cot\tfrac{1}{2}E^\circ = \frac{\cot\frac{1}{2}a\cot\frac{1}{2}b+\cos C}{\sin C}$$

Ex. By these formulas find the spherical excess of some of the triangles referred to in Ex. 1, Art. 56.

59. ***a.*** **The number of spherical degrees in any figure on a sphere,** whatever may be its boundary, is the ratio of the area of the figure to the area of a spherical degree, that is, to $(\frac{1}{360})$th part of the area of the hemisphere (Art. 55). Thus, on a sphere of radius r, if A denotes the area of the figure, and E the number of spherical degrees therein, then, since area of a hemisphere $= 2\pi r^2$,

$$E = A : \frac{1}{360} \text{ of } 2\pi r^2 = \frac{180\,A}{\pi r^2}. \qquad (1)$$

[Compare Art. 57 (1), Art. 59 (2).]

The plane angle E° may be called the *spherical excess of the figure.* For example, the spherical excess of a lune of angle A° is $2A^\circ$.

b. **The spherical excess of a** (*non-re-entrant*) **spherical polygon.** On drawing diagonals from any vertex of a polygon of n sides to the other vertices, it will be seen that the polygon is divided into $n-2$ triangles. The sum of the angles of all these triangles is the same as the sum of the angles of the polygon. Hence,

spherical excess (E°) *of polygon of n sides*

$$= \textit{sum of angles} - (n-2)180^\circ.$$

If the radius of the sphere is r, then (Art. 57)

$$\textit{area of the polygon} = \frac{E}{180}\pi r^2. \qquad (2)$$

60. Given the area of a figure: to find its spherical excess. More fully: *To find the spherical excess of a figure on a sphere when the area of the figure is given in square units.*

Let r denote the radius of the sphere, A the area of the figure, E the number of degrees, n the number of seconds, and E_r the number of radians, in its spherical excess. Then, by (1) Art. 59,

$$E = \frac{180\,A}{\pi r^2}. \tag{1}$$

$$\therefore\ n = 3600\,E = 206265\frac{A}{r^2}. \tag{2}$$

Now $$1^\circ = \frac{\pi}{180} \text{ radians};$$

hence $$E^\circ = \frac{\pi}{180}E \text{ radians}$$

$$= \frac{A}{r^2} \text{ radians.} \qquad [\text{by (1)}]$$

$$\therefore\ E_r = \frac{A}{r^2}. \tag{3}$$

A particular application of (2) can be made to the following problem, viz.: *The area of a spherical triangle on the earth's surface being known, to derive a formula for computing the spherical excess.*

The length of a degree on the earth's surface is found to be 365155 feet. Accordingly,

$$R \text{ (the radius of the earth)} = \frac{365155 \times 180}{\pi} \text{ feet.} \tag{4}$$

From (2), $$\log n = \log A + \log 206265 - 2\log R. \tag{5}$$

On expressing A in square feet, and substituting in (5) the value of R in (4), there is obtained,

$$\log n = \log A - 9.3267737. \tag{6}$$

Formula (6) is called *Roy's Rule*, as it was used by General William Roy (1726–1790) in the Trigonometrical Survey of the British Isles.* The area of the spherical triangle can be approximately determined to a sufficient degree of accuracy.

* The rule should probably be credited to Isaac Dalby (1744–1824), who was mathematical assistant to General Roy from 1787 to 1790, and later became professor of mathematics at the Royal Military College. [See *Phil. Trans.*, vol. 80 (1790).] This was the first practical application of Gerard's theorem (Art. 57).

61. The measure of a solid angle. A *plane angle* can be measured by *any circular arc which it subtends;* and the measure can be expressed in *radians* and in *degrees.* The *radian* (or circular) *measure of an angle* is the number of times any circular arc subtended by it contains the radius (*Plane Trig.*, Art. 73); and the *number of degrees in the angle* is equal to the number of degrees in the subtended circular arc. Thus, the radian measure of an angle of an equiangular triangle is $\frac{1}{3}\pi$, and its degree measure is 60.

A *solid angle* can be measured in a somewhat similar manner, namely, by means of *any spherical surface which it subtends.* What may be called the **spherical measure** *of a solid angle* is the number of times any spherical surface subtended by it contains an area equal to the square on the radius. For example, since the surface of a sphere is equal to $4\pi r^2$, the sum of all the solid angles about any point is 4π. The angle at the corner of a cube subtends one-eighth of the surface of the sphere; accordingly, its spherical measure is $\frac{4\pi r^2}{8} \div r^2$, *i.e.* $\frac{1}{2}\pi$.

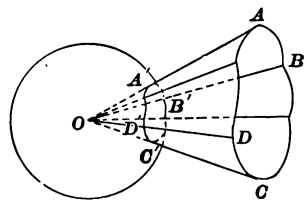

FIG. 50

A solid angle may also be measured in *spherical degrees,* a term that will be explained presently. What may be called the **spherical degree measure** *of a solid angle* (or, the number of spherical degrees in the angle) is a number equal to the number of spherical degrees of area in any spherical surface subtended by the angle. An angle that subtends a spherical degree of surface, contains what may be called *a solid spherical degree.* For example, the sum of all the solid angles about any point is 720 spherical degrees (of angle); the angle at the corner of a cube contains 90 spherical degrees (of angle). Thus the *spherical measure* of the angle at the corner of a cube is $\frac{1}{2}\pi$, and its *spherical degree measure* is 90. On comparing these definitions of solid angular measures with Art. 55 and equations (3) and (1) Art. 60, it is seen that these *measures of solid angles are equal to the measures, in radians and degrees respectively, of the spherical excess of the figures subtended on any sphere by the angle,* when the vertex of the angle is at the centre of the sphere.

NOTE 1. **The term degree.** In geometry and trigonometry the word *degree* is used in connection with *four* very different kinds of quantities; namely, *circular arcs, plane angles, spherical surfaces,* and *solid angles.*

A *degree of arc*, or *an arcual degree*, is $(\frac{1}{360})$th part of any circle;

A *degree of angle*, or *an angular degree*, is $(\frac{1}{360})$th part of four right angles;

A *degree of surface on a sphere*, or a *spherical degree of surface*, is $(\frac{1}{720})$th part of the surface of any sphere;

A *degree of solid angle*, or a *solid spherical degree*, is $(\frac{1}{720})$th part of the solid angles about any point.

NOTE 2. If two plane angles are equal, they can be superposed, the one on the other. On the other hand, just as two figures on a sphere may be equal in area and differ in every other respect, so two solid angles can be equal in measure and differ in every other respect.

NOTE 3. The following remarks relating to the measurement of solid angles are from Hutton's *Course in Mathematics*, Vol. II., p. 64:

"*Solid angles:* If about the angular point of a solid angle as centre, a sphere be described to radius unity, the portion of its surface intercepted between the planes which contain the solid angle is the *measure of the solid angle.* (This method of estimating the magnitude of solid angles appears to have been first given by Albert Girard in his *Invention Nouvelle en Algebre*, 1629; and it would very naturally suggest itself as one of the simplest applications of his theorem for the spherical excess.)" [Compare *Plane Trigonometry*, p. 126, Note 2.]

Ex. 1. The edge angles of a triedral angle are 74° 40′, 67° 30′, 49° 50′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 43.)

Ex. 2. The face angles of a triedral angle are 47° 30′, 55° 40′, 60° 10′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 42.)

Ex. 3. Two face angles of a triedral angle are 64° 24′, 42° 30′, and the edge angle between their planes is 58° 40′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 44.)

Ex. 4. A face angle of a triedral angle is 74° 20′, and the two adjacent edge angles are 67° 30′ and 45° 50′; calculate its measure. (See Ex. 1, Art. 45.)

Ex. 5. Calculate the spherical degree measure, and the spherical measure, of the solid angles corresponding to the spherical triangles described in Art. 42, Exs. 2, 3; Art. 43, Ex. 2; Art. 44. Exs. 2, 3; Art. 45, Ex. 2; Art 46, Exs. 2, 3; Art. 47, Ex. 2. (See Ex., Art. 56.)

62. The volume of a sphere. In some works on solid geometry and in books on mensuration it is shown that the volume of a pyramid is equal to one third the product of its base and altitude. Now suppose that a polyedron (*i.e.* a solid bounded by plane faces) is circumscribed about a sphere, each of the faces of the polyedron, accordingly, touching the sphere. This polyedron may be regarded as made up of pyramids which have a common vertex (namely, the centre of the sphere), and a common altitude (namely, the radius of the sphere), and which have the faces of the polyedron as bases. Then, R being the radius of the sphere,

$$\text{Vol. of polyedron} = \tfrac{1}{3} R \times (\text{sum of faces of polyedron}). \quad (1)$$

If the number of faces of the polyedron be increased and the area of each face be decreased, then the sum of the faces becomes more nearly equal to the area of the surface of the sphere, and the volume of the polyedron becomes more nearly equal to the volume of the sphere. By increasing the number of faces and decreasing the area of each face, the difference between the sum of the faces of the polyedron and the area of the sphere can be made as small as one please; and, likewise, the difference between the volume of the polyedron and the volume of the sphere can be made as small as one please. In other words:

The area of the surface of the sphere is the limit of the area of the surface of the polyedron, and the volume of the sphere is the limit of the volume of the polyedron, when the faces of the latter are increased without limit, and each face is made to approach zero in area.

Hence, from (1), Vol. of sphere $= \tfrac{1}{3} R \times$ surface of sphere

$$= \tfrac{4}{3} \pi R^3.* \quad (2)$$

63. Definitions. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of the sides of the polygon. The polygon is called the *base* of the pyramid.

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* For a note concerning the measurement of the circle and the sphere see *Plane Trigonometry*, Art. 72, and Note *C*, p. 171. For the proofs of Archimedes, see T. L. Heath, *The Works of Archimedes edited in modern notation, with introductory chapters* (Cambridge, University Press), pp. 39, 41, 93.

For example, in Fig. 11, Art. 12, *O–ABCD*, *O–ABC*, *O–ABD*, are spherical pyramids; their bases are *ABCD*, *ABC*, *ABD*.

A **spherical sector** is the portion of a sphere generated by the revolution of a sector of a circle about any diameter of the circle as axis. For example, in Fig. 47, Art. 53, when the semicircle *ATB* revolves about *AB*, each of the circular sectors *AOL*, *LOT*, *LOK*, etc., describes a spherical sector.

A **spherical segment** is the portion of a sphere bounded by two parallel planes and the zone intercepted between them. (One of the planes may be tangent to the sphere.)

64. Volume of a spherical pyramid; of a spherical sector. By reasoning analogous to that in Art. 62, it can be shown that, in a sphere of radius R,

$$\text{vol. of a spherical pyramid} = \tfrac{1}{3} R \times \text{area of its base};$$

$$\text{vol. of a spherical sector} = \tfrac{1}{3} R \times \text{area of its zone.}$$

Since the area of a zone of height $h = 2\pi Rh$ (Art. 53),

then $$\text{vol. of spherical sector} = \tfrac{2}{3}\pi R^2 h.$$

Thus in Fig. 11, Art. 12,

$$\text{vol. } O\text{–}ABCD = \tfrac{1}{3}\, OA \times \text{area } ABCD;$$

in Fig. 47, Art. 53,

vol. of sector described by $AOL = \tfrac{1}{3}\, OA \times$ area of zone described by arc $AL = \tfrac{2}{3}\pi R^2 \cdot Al$, and

vol. of sector described by $LOT = \tfrac{1}{3}\, OA \times$ area of zone described by arc $LT = \tfrac{2}{3}\pi R^2 \cdot lt$.

EXAMPLES.

1. Find the volumes of the spherical pyramids whose bases are the triangles described in Art. 57, Exs. 1–6.

2. Find the volumes of the following spherical sectors:

(*a*) The sector whose base is a zone of height 2 inches on a sphere of radius 18 inches.

(*b*) The sector whose base is a zone of height 3 feet on a sphere of radius 12 feet.

65. Volume of a spherical segment. Let AB be an arc of a semicircle of radius R having the diameter DD'. From A, B, draw Aa, Bb, at right angles to DD'. It is required to find the volume of the spherical segment generated by the revolution of $ABba$ about DD'.

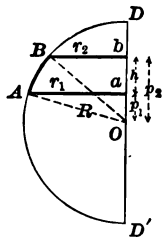


FIG. 51

Let h denote the height of the segment, and p_1, p_2 the lengths of the perpendiculars from the centre O to the parallel bases of the segment. On making the revolution of the semicircle DAD', it is seen that

segment generated by $ABba$ = cone generated by BOb + spherical sector generated by AOB − cone generated by AOa.

Now, vol. cone generated by $BOb = \frac{1}{3}\pi r_2^2 p_2$;

vol. sector generated by $AOB = \frac{2}{3}\pi R^2 h$; (Art. 64)

vol. cone generated by $AOa = \frac{1}{3}\pi r_1^2 p_1$.

$$\therefore \text{ vol segment} = \tfrac{1}{3}\pi(r_2^2 p_2 + 2R^2 h - r_1^2 p_1). \qquad (1)$$

NOTE. The result (1) can be reduced to various forms. For example, since

$$p_1^2 = R^2 - r_1^2,\ p_2^2 = R^2 - r_2^2,\ p_2 - p_1 = h,$$

then vol. segment $= \frac{2}{3}\pi R^2(p_2 - p_1) + \frac{1}{3}\pi p_2(R^2 - p_2^2) - \frac{1}{3}\pi p_1(R^2 - p_1^2)$

$$= (p_2 - p_1)\pi R^2 - \tfrac{1}{3}\pi(p_2^3 - p_1^3) \qquad (2)$$

$$= \frac{p_2 - p_1}{3}\pi[3R^2 - (p_2^2 + p_2 p_1 + p_1^2)]. \qquad (3)$$

Since $h = p_2 - p_1$, then $h^2 = p_2^2 - 2p_2 p_1 + p_1^2$.

$$\therefore p_1 p_2 = \frac{p_2^2 + p_1^2 - h^2}{2}, \text{ and } p_2^2 + p_2 p_1 + p_1^2 = \frac{3}{2}(p_1^2 + p_2^2) - \frac{h^2}{2}.$$

On substituting the last result in (3), expressing p_1^2 and p_2^2 in terms of R, r_1, r_2, and reducing, the following formula is obtained, viz.:

$$\text{vol. segment} = \frac{\pi h}{2}\left(r_1^2 + r_2^2 + \frac{h^2}{3}\right). \qquad (4)$$

EXAMPLES.

1. Show that if (in Fig. 51) angle $AOD = \alpha$, then the volume of the spherical sector generated by AOD is $\frac{2}{3}\pi R^3(1 - \cos\alpha)$.

2. Show that if angle $AOD = \alpha$, then the volume of the segment generated by the revolution of ADa is $\frac{1}{3}\pi R^3 \sin^4\frac{1}{2}\alpha(1 + 2\cos^2\frac{1}{2}\alpha)$.

SUGGESTION. Segment generated by ADa = sector generated by AOD − cone generated by AOa.

3. Find the volume of a spherical segment, the diameters of its ends being 10 and 12 inches, and its height 2 inches.

4. The diameters of the ends of a spherical segment are 8 and 12 inches, and its height is 10 inches. Find its volume.

N.B. *For questions and exercises on Chapter VI., see page* 108.

CHAPTER VII.

PRACTICAL APPLICATIONS.

66. Geographical problem. *To find the distance between two places and the bearing* (*i.e.* the direction) *of each from the other, when their latitudes and longitudes are known.* An interesting application of spherical trigonometry can be made in solving this problem. In the following examples the earth is regarded as spherical, and its radius is taken to be 3960 miles.

EXAMPLES.

1. Find the shortest distance along the earth's surface between Baltimore (lat. 39° 17′ N., long. 76° 37′ W.) and Cape Town (lat. 33° 56′ S., long. 18° 26′ E.).

In Fig. 52 B and C represent Baltimore and Cape Town; EQ is the earth's equator; NGS, NBS, NCS are the meridians of Greenwich, Baltimore, and Cape Town respectively; BC is the great circle arc whose length is required.

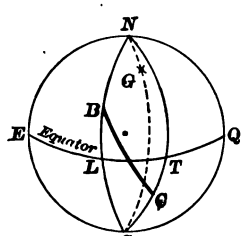


FIG. 52

In the spherical triangle BNC, NB, NC, and BNC are known. For

$$NB = 90° - BL = 90° - 39° 17′ = 50° 43′$$
$$NC = 90° + TC = 90° + 33° 56′ = 123° 56′$$
$$BNC = BNG + GNC = 76° 37′ + 18° 26′ = 95° 3′$$

Hence, BC can be determined in degrees by Art. **44**; then, the radius of the sphere being given, BC can be determined in miles. The angles NBC, NCB, can also be found.

Answers: $BC = (65° 47′ 48″) = 4685.8$ miles; $NBC = 115° 1′ 35′$; $NCB = 57° 42′ 23″$.

NOTE 1. The *bearing* of one place from a second place is the angle which the great circle arc joining the two places makes with the meridian of the second place. Thus, in Fig. 52 the bearing of Cape Town from Baltimore is the angle NBC, and the bearing of Baltimore from Cape Town is NCB.

Since $NBC = 115° 1' 35''$ the ship *sets out from Baltimore* on a course S. 64° 58′ 25″ E.; since $NCB = 57° 42' 23''$ the ship *approaches Cape Town* on a course S. 57° 42′ 23″ E.

NOTE 2. A ship that sails on a great circle (excepting the equator or a meridian) must be continually changing her course.

2. Find the latitude of the place where BC crosses the meridian 15° W.; also find the bearing of Cape Town from this place.

3. If a vessel sails from Baltimore and keeps constantly on the course (see Ex. 1) S. 64° 58′ 25″ E. (*i.e.* crosses every meridian at the angle 64° 58′ 25″), will she arrive at Cape Town? [*Answer.* No.]

4. What path will the vessel in Ex. 3 make on the sea? *Answer.* A path which is a spiral going round and round the earth and gradually approaching the south pole. This path is called the *loxodrome*, or *rhumb line.*

5. If a person leaves Boston, Mass. (lat. 42° 21′ N., long. 71° 4′ W.), starting due east, and keeps on a great circle: (*a*) Where will he be after he has passed over an arc of 90°, and in what direction will he be going? (*b*) Where will he be after he has passed over an arc of 180°, and in what direction will he be going? (*c*) Where will he be after he has passed over an arc of 270°, and in what direction will he be going? [Solve this example: (1) by spherical geometry; (2) by spherical trigonometry.]

6. What is the distance from New York (40° 43′ N., 74° 0′ W.) to Liverpool (53° 24′ N., 3° 4′ W.)? Find the bearing of each place from the other. In what latitude will a steamer sailing on a great circle from New York to Liverpool cross the meridian of 50° W.; and what will be her course at that point?

N.B. *Check the results in the following exercises:*

7. Find the distance and bearing of Liverpool from Montreal (45° 30′ N., 73° 33′ W.).

8. Find the distance and bearing of Liverpool from Halifax, N. S. (44° 40′ N., 63° 35′ W.).

9. Find the distance and bearing of Santiago de Cuba (20° N., 75° 50′ W.) from Rio de Janeiro (22° 54′ S., 43° 8′ W.).

10. Find the distance and bearing of San Francisco (37° 47′ 55″ N., 122° 24′ 32″ W.) from New York.

11. Find the distance of Victoria, B. C. (48° 25′ N., 123° 23′ W.) from Sydney, N. S. W. (33° 52′ S., 151° 13′ E.); and the bearing of each place from the other.

12. Find the distances between the following places: (*a*) San Francisco and Honolulu; (*b*) Cape Town and Cairo; (*c*) Honolulu and Manila; (*d*) Victoria, B. C., and Tokio.

13. Find the distances between other places, and their bearings from each other.

APPLICATIONS TO ASTRONOMY.

N.B. *In connection with his study of the following articles the student should consult some elementary text-book on astronomy. The numerical examples given here will supplement his outside reading on spherical astronomy.*

67. One of the most important applications of spherical trigonometry is to astronomy. Trigonometry was invented to aid astronomy, and for centuries was studied as an adjunct of the latter subject. (See *Plane Trigonometry*, pp. 165, 166.) A few of the simplest problems of spherical astronomy are introduced in Arts. 73, 74. In order to understand these problems a clear conception of a few astronomical terms and principles is necessary. These terms are explained in Arts. 68–72.

68. The celestial sphere. To a person on the surface of the earth, the sky above is like a great hemispherical bowl with himself at the centre. The stars seem to move from east to west across the spherical sky in parallel circles whose axis is the earth's polar axis prolonged. Each star makes a complete revolution about this axis in 23 hours 56 minutes ordinary clock time. The stars appear never to change their positions with reference to one another, being in this respect like places on the earth's surface.* Another way of describing the relations of the earth and the enveloping sky, is to say that the whole sky is turning, like an immense crystal sphere, about an axis which is the earth's polar axis prolonged, the motion being from east to west. The stars keep the same positions with respect to one another, and, accordingly, appear to be attached to the surface of the sphere. As the sphere turns, the stars fixed in it appear to trace parallel circles

* The positions of some of the stars suffer a very slight change which is perceptible in the course of centuries.

about the axis. The sphere turns completely in 23 hours 56 minutes ordinary clock time.* The *stars all seem to be at the same distance* from the observer because his eyes can judge their *directions only*, and not their *distances*.

The following considerations will show that it is natural enough for an observer on the earth to think that he is always *at the centre* of the sphere on which the stars appear to be. When a person changes his position, the direction of an object at which he is looking changes also, unless he moves directly towards or away from the object. For instance, from a certain point a tree may be in an easterly direction, and when the observer moves a little way the tree may be in a southeasterly direction. Moreover, the further away an object is, the less will be the change in its direction caused by any particular change in the observer's position. Thus, if a person is near a tree, a few steps on his part may change the direction of the tree from east to southeast, but if he is five miles from the tree, an equal number of steps taken by him will make very little difference in the direction of the tree. Now the earth's mean distance from the sun is about 93,000,000 miles. Hence, an observer who now looks at the stars from a certain position, in about six months from now will look at them from a point 186,000,000 miles distant from his present position.† Astronomers have succeeded in a few instances in determining the distances of the stars from the earth.‡ It has been found that the nearest star yet known, *Alpha Centauri*, is so far away that the *change in its direction* from the centre of the earth, due to the change of position of 186,000,000 miles on the part of the earth, *is less* than the change in the direction of an object 3½ miles away when the observer moves his head a couple of inches at right angles to the line of sight. This being so in the case of the sun's *nearest* stellar neighbour, it is natural for an observer on the earth to think that he is always at the centre of the great sphere on which the stars appear to be; and it is perfectly proper

* The student probably knows that the apparent turning of the spherical sky from east to west about an axis which is the earth's polar axis prolonged, is really due to the rotation of the earth in an opposite direction. The observer is not conscious of any motion of the earth, and thinks that the sky with its bright points is revolving about the earth from east to west, while all the time the sky is motionless, and the earth is turning under it from west to east. Just as to a person in a swiftly moving train the objects outside seem to be rushing by him while the train appears to be at rest.

† This, moreover, does not take any account of the motion of the sun with his system through space.

‡ The first stellar distance determined was that of *61 Cygni* by Friedrich Wilhelm Bessel (1784–1846), one of the greatest of German astronomers, in 1838. Since then the distances of about 100 stars have been measured; about 50 of these distances are regarded as reliably determined.

for him to act in accordance with this notion when he makes astronomical observations and deductions.*

The sphere on which the stars appear to move in parallel circles, or, what comes to the same thing, the sphere which appears to have the stars attached to it and to revolve about the earth's polar axis prolonged, is called **the celestial sphere.**

69. Points and lines of reference on the celestial sphere. There will now be shown some methods for indicating the positions of the heavenly bodies on the celestial sphere — *their positions with respect to the observer* and *their positions with respect to one another.*

The positions of places on the terrestrial sphere are described by means of certain points and great circles on the sphere. There are various pairs of circles which are used for reference; for example, the equator (whose poles are the north and south poles of the earth) and the meridian passing through the Royal Observatory at Greenwich, the equator and the meridian passing through the observatory at Washington, etc. It will be observed that in each case *the reference circles are at right angles to each other, and, accordingly, each of them passes through the poles of the other.*

In an analogous manner the positions of bodies on the celestial sphere are described by means of, or by reference to, certain points and great circles on that sphere. There are *four* different systems of circles of reference. As in the case of the terrestrial sphere, each system consists of two circles, each of which passes through the pole of the other, and, accordingly, is at right angles to the other. Two of these systems are described in Arts. 70, 71, a third in Art. 76, and the fourth in Art. 77. A point which will be referred to in these systems is the **north celestial pole.** This is the point where the earth's axis, if prolonged, would pierce the celestial sphere. It is near the pole star, being about $1\frac{1}{4}°$ from it.

* ". . . imagine the entire solar system as represented by a tiny circle the size of the dot over this letter *i*." (Neptune the outermost planet known of the solar system is 2790 millions of miles from the sun; *i.e.* 30 times as far as the earth.) "Even the sun itself, on this exceedingly reduced scale, could not be detected with the most powerful microscope ever made. But on the same scale the vast circle centred at the sun and reaching to *Alpha Centauri* would be represented by the largest circle which could be drawn on the floor of a room 10 feet square." (Todd, *New Astronomy*, p. 438.)

70. The horizon system: Positions described by altitude and azimuth. For any place on the earth's surface, the point at which the plumb line extended upwards meets the celestial sphere is called **the zenith**; the diametrically opposite point is called **the nadir**. If a plane perpendicular to the plumb line be passed either immediately beneath the observer's feet, or through the centre of the earth, about 4000 miles below him, then the intersection of this plane with the celestial sphere is called **the horizon**. (Since the earth is so small and so far away from even the nearest star, two parallel planes 4000 miles apart and passing through the earth will appear, to a terrestrial observer, to intersect the celestial sphere in the same great circle.)

Great circles passing through the zenith are perpendicular to the horizon; they are called *vertical circles*. The *north point of the horizon* is the point which is directly north from the observer. It is where the vertical circle passing through the north pole intersects the horizon. *This circle which passes through the zenith and the pole is called* **the meridian** *of the observer. The horizon and the meridian* are the reference circles in the horizon system

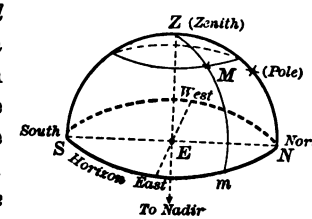


FIG. 53

The **altitude** (denoted by h) of a heavenly body is its angular distance above the horizon. Thus the altitude of M (Fig. 53, in which E is the earth and Z the zenith of the place of observation) is Mm. The altitude of the zenith is 90°. The distance of a star from the zenith is called its *zenith distance*; this is obviously the complement of the altitude.

The **azimuth** (denoted by A) of a heavenly body is the angle between its vertical circle and the meridian. This angle is measured usually along the horizon from the south point in the direction of the west point, to the foot of the star's vertical circle. Thus in Fig. 53 the azimuth of M is $180° + NZm$, which is measured by the arc $180° + Nm$ on the horizon.

NOTE. Any two points on the earth's surface have different zeniths. Hence, the above system of describing positions on the celestial sphere is peculiarly *local*. Moreover, a star rises in the eastern part of the horizon

(altitude zero), mounts higher in the sky until it reaches the observer's meridian, then sinks towards, and sets in, the west; it is, accordingly, continually changing its altitude and azimuth.

71. The equator system: Positions described by declination and hour angle. The north celestial pole is the principal point of this system. **The celestial equator** is the great circle of which that point is the pole; it is evidently the projection of the earth's equator upon the celestial sphere. *The celestial equator and the meridian of the observer* are the reference circles in the system now being described. In Fig. 54, P is the north celestial pole, S the south celestial pole, EQ the celestial equator; also, HR is the horizon and Z the zenith for some particular place on the earth's surface. As said in Art. 68, the stars move in parallel circles whose axis is PS; these circles are, accordingly, parallel to the equator EQ. The angular distance of a star from the equator is called the **declination** (denoted by D or δ) of the star; *north* (or +) *declination* when the star is north of the equator, and *south* (or −) *declination* when the star is south. Thus the declination of S_3 is S_3s_3. The angular distance of a star from the north pole is called its *north polar distance;* this is evidently the complement of the star's declination.*

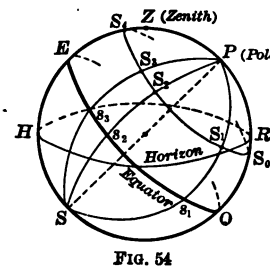


FIG. 54

In 24 (sidereal) hours a star appears to make a complete revolution (*i.e.* to pass over 360°) about the celestial polar axis; hence, *the star passes over 15° in 1 hour.*† The great circles passing through the poles are called *hour circles.* Thus PS_3S is the hour circle of S_3. **The hour angle** (denoted by *H. A.*) of a star is the angle between the meridian of the observer and the hour circle of the

* The declination of the stars change by an exceedingly small amount in the course of a year.

† The interval of time between two successive passages of the observer's meridian by the sun (*i.e.* from noon to noon) is about 4 minutes longer than the interval of time between two successive passages of the meridian by any particular star. (This difference is due to the yearly revolution of the earth about the sun. See text-books on astronomy.) The second interval is called a sidereal day; it is divided into 24 sidereal hours.

The right ascension (denoted by R.A.) of a heavenly body is the angle at the north celestial pole between the hour circle of the body and the hour circle of the vernal equinox. This angle is measured from the latter circle *towards the east*, from 0° to 360° or 1 h. to 24 h.; it may be measured by the arc intercepted on the equator. Declination has been defined in Art. 71.

In Fig. 58, P is the north celestial pole, E_2Q the equator, E_1C the ecliptic, and V the vernal equinox. If S is any star, then for S $D = SM$, and R.A. = angle VPM = arc VM.

77. The ecliptic system: Positions described by latitude and longitude. In this system the point and circles of reference are *the pole of the ecliptic, the ecliptic,* and *the great circle passing through the pole of the ecliptic and the vernal equinox.* The **latitude** of a star is its angular (or arcual) distance from the ecliptic; its **longitude** is the angle at the pole of the ecliptic between the circle passing through this pole and the vernal equinox and the circle passing through this pole and the star. This angle may be measured by the arc intercepted on the ecliptic. It is always measured *towards the east* from the vernal equinox.

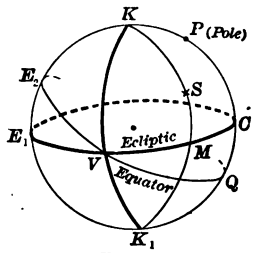


FIG. 59

In Fig. 59, K is the pole of the ecliptic, E_1C the ecliptic, P the pole of the equator, E_2Q the equator, and V the vernal equinox. If S is any star, then

$$\text{latitude of } S = SM, \text{ longitude of } S = VKM = VM.$$

When the latitude and longitude of a star are known, its declination and right ascension can be found, and vice versa. For, in the triangle KPS (the triangle whose vertices are the star and the poles of the equator and the ecliptic), $KP = 23\frac{1}{2}°$ (since $QVC = 23\frac{1}{2}°$), $KS = 90° - \text{lat.}$, $SKP = 90° - \text{long.}$, $SP = 90° - D$; $SPK = VPK - VPS = 90° - (360° - \text{R.A.})$, if S is west of VP; $SPK = 90° + \text{R.A.}$, if S is east of VP. If any two of these be known besides KP, the remaining two can be found by solving KPS.

N.B. *Questions and exercises on Chapter VII. will be found at page* 109.

APPENDIX.

NOTE A.

ON THE FUNDAMENTAL FORMULAS OF SPHERICAL TRIGONOMETRY.

1. The relations between the sides and angles of a right-angled spherical triangle were obtained in Art. 26. The law of sines and the law of cosines (Art. 36) for any spherical triangle have been derived by means of these relations. (See Note 1, Art. 36.) These two laws can also be derived directly by geometry; this is done in Arts. 2, 3, below. Moreover, the law of sines can be derived analytically from the law of cosines, as shown in Art. 4. In Art. 5 it is shown how the relations for right-angled triangles can be derived from these two laws. Other relations between the parts of a spherical triangle have been referred to in Art. 40; these relations can also be deduced by means of the law of cosines and the law of sines. *The law of cosines is, accordingly, the fundamental and most important formula in spherical trigonometry.*

2. Direct geometrical derivation of the law of cosines. Let O–ABC be a triedral angle, and ABC be the corresponding spherical triangle on a sphere of radius OA. It is required to find the cosine of the face angle COB, or, what is the same thing, the cosine of the side CB.

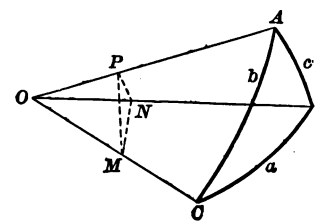


FIG. 60

In OA take any point P, and through P pass a plane MPN at right angles to the line OA. Then OPN and OPM are right angles, and angle MPN = angle A. Also, the measures (in degrees) of the sides AB, BC, CA, are the same as the measures of the face angles COB, BOA, AOC, respectively.

In MPN, $\quad \overline{MN}^2 = \overline{MP}^2 + \overline{PN}^2 - 2\,MP \cdot PN \cos MPN$; (1)

In MON, $\quad \overline{MN}^2 = \overline{MO}^2 + \overline{ON}^2 - 2\,MO \cdot ON \cos MON$. (2)

Hence, on equating these values of $\overline{MN}^2$ and transposing,

$$2\,MO \cdot ON \cos MON = \overline{MO}^2 - \overline{MP}^2 + \overline{ON}^2 - \overline{PN}^2 + 2\,MP \cdot PN \cos MPN.$$

Now $\overline{OM}^2 - \overline{MP}^2 = \overline{OP}^2$, and $\overline{ON}^2 - \overline{PN}^2 = \overline{OP}^2$, since OPM and OPN are right angles.

$$\therefore 2\,MO \cdot ON \cos MON = 2\,OP^2 + 2\,MP \cdot PN \cos MPN.$$

$$\therefore \cos MON = \frac{OP}{MO}\frac{OP}{ON} + \frac{MP}{MO}\frac{PN}{ON}\cos MPN;$$

i.e. $$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (3)$$

Like formulas for $\cos b$, $\cos c$, can be derived in a similar manner; they can also be written immediately, on paying regard to the symmetry in (3). The formulas for $\cos A$, $\cos B$, and $\cos C$, can be derived by means of the polar triangle, as done in Art. 36, *C*.

EXERCISES.

1. Make the figure and derive the law of cosines: (*a*) when P is taken at A; (*b*) when P is taken in OA produced towards A.

2. Derive the formula for $\cos b$ geometrically. (Take any point in OB, and through this point pass a plane at right angles to OB.)

3. Derive the formula for $\cos c$ geometrically

3. Direct geometrical derivation of the law of sines. Let O–ABC be a triedral angle, and ABC be the corresponding spherical triangle on a sphere of radius OA.

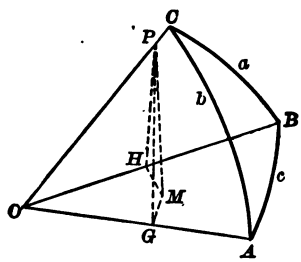


FIG. 61

In OC take any point P, and draw PM at right angles to the plane AOB, and intersecting this plane in M. Through M draw MG and MH, at right angles to OA and OB respectively. Pass a plane through the lines PM and MG.

Since PM is perpendicular to OAB, the plane PMG is perpendicular to OAB (Euc. XI. 18). Hence, since AGM is a right angle, AGP is also a right angle. Therefore angle PGM = angle A. Similarly it can be shown that angle PHM = angle B.

$$\therefore \sin A = \frac{PM}{PG} = \frac{PM}{OP \sin AOC} = \frac{PM}{OP \sin b}. \quad \therefore \sin A \sin b = \frac{PM}{OP}. \quad (1)$$

Also, $\sin B = \frac{PM}{PH} = \frac{PM}{OP \sin BOC} = \frac{PM}{OP \sin a}$. $\therefore \sin B \sin a = \frac{PM}{OP}$ (2)

$\therefore$ by (1), (2), $\sin A \sin b = \sin B \sin a$.

$\therefore$ $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}$.

In a similar way it can be shown that $\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}$. Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Ex. 1. Show geometrically:

(a) that $\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}$; ($b$) that $\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$.

Ex. 2. Make the derivation when M is not in the sector AOB.

4. Analytical derivation of the law of sines from the law of cosines.

$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$. [From (3) Art. 2]

$$\therefore 1 - \cos^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

$$= \frac{(1-\cos^2 b)(1-\cos^2 c) - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

i.e. $\sin^2 A = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}$.

$$\therefore \frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}. \quad (1)$$

Similarly, $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ can each be shown to be equal to the second member of (1). Hence,

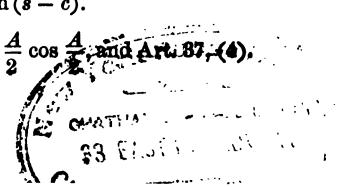
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}; \quad (2)$$

in which $2n$ denotes the positive square root of the numerator of the second member of (1).

Ex. 1. Show the truth of the statement made above.

Ex. 2. Show that the numerator in the second member of (1) is equal to $4 \sin s \sin (s-a) \sin (s-b) \sin (s-c)$.

SUGGESTION. $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$, and Art. 37, (4).



5. Formulas for right-angled triangles derived from the general formulas. In the triangle ABC let angle $C = 90°$. Then $\sin C = 1$, and relations (1), p. 45, become (2) and (2′), p. 30. Also, $\cos C = 0$, and the third formula in Art. 36, B becomes (1), p. 30. The three formulas in Art. 36, C reduce to (5), (5′), and (6), p. 30, respectively. Formulas (3), (3′), (4) and (4′), p. 30, can be derived from the others on that page. For

$$\cos A = \sin B \cos a \text{ [by (5′)]} = \frac{\sin b}{\sin c} \cdot \frac{\cos c}{\cos b} \text{ [by (2′), (1)]} = \frac{\tan b}{\tan c};$$

similarly,
$$\cos B = \frac{\tan a}{\tan c}.$$

Also,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sin B \cos a} \text{ [by (5′)]} = \frac{\sin a}{\sin b \cos a} \text{ [by (2), (2′)]} = \frac{\tan a}{\sin b};$$

similarly,
$$\tan B = \frac{\tan b}{\sin a}.$$

Other relations in triangles (see Art. 40) can also be used in the derivation of the formulas for right-angled triangles.

EXERCISES.

1. Deduce the law of cosines: (1) directly, by geometry; (2) by means of the relations in a right-angled triangle.

2. Deduce the law of sines: (1) analytically, from the law of cosines (2) directly, by geometry; (3) by means of the relations in a right-angled triangle.

3. Deduce the ten relations between the sides and angles of a right-angled spherical triangle: (1) by means of the relations between the sides and angles of the general spherical triangle; (2) directly, by geometry.

NOTE B.

[Supplementary to Art. 58.]

DERIVATION OF FORMULAS FOR THE SPHERICAL EXCESS OF A TRIANGLE.

***I.* Cagnoli's Formula.** (*In terms of the sides.*)

$$\sin \tfrac{1}{2}E = \sin \tfrac{1}{2}(A + B + C - 180°) = -\cos \tfrac{1}{2}(A + B + C)$$
$$= \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}C - \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}C$$
$$= \frac{\sin \tfrac{1}{2}C \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}c} [\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}(a + b)] \quad \text{[Art. 39, (1), (3)]}$$
$$= \frac{\sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin C}{\cos \tfrac{1}{2}c} \quad \text{[Arts. 50 (5), 52 (8), } \textit{Plane Trig.}\text{]} \quad (1)$$
$$= \frac{\sin \tfrac{1}{2}a \sin \tfrac{1}{2}b}{\cos \tfrac{1}{2}c} \cdot \frac{2n}{\sin a \sin b} \quad \text{[Note A, Art. 4, Eq. (2)]}$$
$$= \frac{n}{2 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}.$$

II. **Lhuillier's Formula.** (*In terms of the sides.*)

$$\tan \tfrac{1}{4} E = \frac{\sin \tfrac{1}{2}(A+B+C-180°)}{\cos \tfrac{1}{2}(A+B+C-180°)}$$

$$= \frac{\sin \tfrac{1}{2}(A+B) - \sin \tfrac{1}{2}(180° - C)}{\cos \tfrac{1}{2}(A+B) + \cos \tfrac{1}{2}(180° - C)} \qquad [*Plane\ Trig.*, \text{p. } 94]$$

$$= \frac{\sin \tfrac{1}{2}(A+B) - \cos \tfrac{1}{2} C}{\cos \tfrac{1}{2}(A+B) + \sin \tfrac{1}{2} C}$$

$$= \frac{\cos \tfrac{1}{2}(a-b) - \cos \tfrac{1}{2} c}{\cos \tfrac{1}{2}(a+b) + \cos \tfrac{1}{2} c} \cdot \frac{\cos \tfrac{1}{2} C}{\sin \tfrac{1}{2} C} \qquad [\text{Art. } 39, (1), (3)]$$

$$= \frac{\sin \tfrac{1}{2}(s-b) \sin \tfrac{1}{2}(s-a)}{\cos \tfrac{1}{2} s \cos \tfrac{1}{2}(s-c)} \sqrt{\frac{\sin s \sin (s-c)}{\sin (s-a) \sin (s-b)}}$$

[Art. 37, (6); *Plane Trig.*, p. 94]

$$= \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

III. **Formula in terms of two sides and their included angle.**

$$\cos \tfrac{1}{2} E = \cos \tfrac{1}{2}(A+B+C-180°) = \sin \tfrac{1}{2}(A+B+C)$$

$$= \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2} C + \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} C$$

$$= [\cos \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2} C + \cos \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2} C] \sec \tfrac{1}{2} c$$

[Art. 39, (1), (3)]

$$= (\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C) \sec \tfrac{1}{2} c. \qquad (2)$$

Hence, from (1) and (2), on division and reduction,

$$\tan \tfrac{1}{2} E = \frac{\tan \tfrac{1}{2} a \tan \tfrac{1}{2} b \sin C}{1 + \tan \tfrac{1}{2} a \tan \tfrac{1}{2} b \cos C}.$$

On taking the reciprocals and reducing, this takes the form

$$\cot \tfrac{1}{2} E = \frac{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b + \cos C}{\sin C}.$$

QUESTIONS AND EXERCISES FOR PRACTICE AND REVIEW.

CHAPTER I.

1. On a sphere let N be the pole of a great circle ABC, and P be any point on the surface between N and ABC; also let $DPNG$ be a semicircle drawn through P at right angles to ABC, and let it intersect ABC in D and G: prove (a) that PD is the shortest great-circle arc that can be drawn from P to ABC; (b) that PNG is the longest great-circle arc that can be drawn from P to ABC.

2. Show that the greater the distance of the plane of a small circle from the centre of the sphere, the less is the circle.

3. The radius of a sphere is 10 inches, and the radius of a small circle upon it is 6 inches. Find: (a) the distance between the centre of the sphere and the centre of the small circle; (b) the angular radius of the small circle; (c) the polar distance (or arcual radius) of the small circle; (d) the distance on the sphere from the small circle to the great circle having the same axis.

4. Prove that if a spherical triangle has two right angles, the sides opposite them are quadrants, and the third angle has the same measure as its opposite side.

5. Prove that in any spherical right triangle an angle and its opposite side are always in the same quadrant.

6. Prove that any side of a spherical triangle is greater than the difference between the other two sides.

7. Prove that each angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

8. Show that the surface of a sphere is eight times the surface of a trirectangular triangle.

9. (a) Show that a trirectangular triangle is its own polar; (b) show that a triquadrantal triangle is its own polar.

10. Show that if two great circles are equally inclined to a third, their poles are equidistant from the pole of the third.

11. Show that the arc through the poles of two great circles cuts both circles at right angles.

12. A ship sails along the parallel of 45° N. a distance of 600 nautical miles. Find the difference of longitude that she has made.

13. Two places in latitude 60° N. are 150 statute miles apart. Find their difference of longitude. [Take the radius of the earth as 3960 miles.]

14. Compare the lengths of the parallels of 30° N., 45° N., and 60° N., with the length of the equator.

15. Prove that if the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.

16. Show that in two polar triangles each angle of the one is the supplement of the side opposite to it in the other.

17. Show that the sum of the angles of a spherical triangle is greater than two, and less than six, right angles.

18. Discuss the following cases, in which A, a, and b are given in a spherical triangle ABC:

I. $A = 90°$: (1) $b = 90°$; (2) $b < 90°$($a < b$, $a = b$, $a > b$ and $< \pi - b$, $a = \pi - b$, $a > \pi - b$); (3) $b > 90°$($a < \pi - b$, $a = \pi - b$, $a > \pi - b$ and $< b$, $a = b$, $a > b$).

II. $A < 90°$: (1) $b = 90°$($a < A$, $a = A$, $a > A$ and $< b$, $a = b = 90°$, $a > b$); (2) $b < 90°$($a < p$, $a = p$, $a > p$ and $< b$, $a = b$, $a > b$ and $< \pi - b$, $a = \pi - b$, $a > \pi - b$); (3) $b > 90°$($a < p$, $a = p$, $a > p$ and $< \pi - b$, $a = \pi - b$, $a > \pi - b$ and $< b$, $a = b$, $a > b$). [For definition of p, see p. 26.]

III. $A > 90°$: (1) $b = 90°$($a = b$, a between b and $\pi - b$, a between $\pi - b$ and p); (2) $b < 90°$($a > p$, $a = p$, $a < p$ and $> b$, $a = b$, a between b and $\pi - b$); (3) $b > 90°$($a < b$, $a > b$ and $< p$, $a < p$ and $> \pi - b$, a between b and $\pi - b$, $a = b$).

CHAPTER II.

1. Define *spherical angle, spherical triangle, Napier's circular parts, polar triangle, quadrantal triangle, oblique spherical triangle, pole of an arc, spherical excess, spherical polygon.*

2. In a right-angled spherical triangle show that: (a) It is impossible for only one of the three sides to be greater than 90°; (b) The hypotenuse is less than 90° only when both the other sides are in the same quadrant; (c) If another part besides the right angle be right, the triangle is biquadrantal.

3. Prove, by geometry and by trigonometry, that in a right spherical triangle an angle and its opposite side are always in the same quadrant, that is, either both are less or both are greater than 90°.

4. Prove that in a right spherical triangle ABC, $(C = 90°)$. (a) $\sin A = \cos B \div \cos b$; (b) $\cos c = \cot A \cot B$; (c) $\cos c = \cos a \cos b$.

5. (a) Mention in order Napier's circular parts, and state the two principal rules for their use. (b) State Napier's Rules and write the ten formulas for the right spherical triangle by means of them. (c) Prove three of these formulas.

6. What formulas should be used to find B, a, and b of a right spherical triangle ABC $(C = 90°)$ when A and c are given? What formula includes all the required parts?

7. Show how to obtain the formulas for finding a, B, and C of a quadrantal triangle, when A and b are given and $c = 90°$.

8. Given one side and the hypotenuse of a right spherical triangle, write all the formulas for the solution and check, and state how the species of each part will be determined.

9. How many solutions are there for a right spherical triangle ABC, given side b and angle B? Discuss fully.

10. Given A and b of a right spherical triangle ABC $(C = 90°)$: write and derive formulas for computing each of the parts B, a, and c in terms of A and b only; also the check formula.

11. Show how to solve a right spherical triangle, having given (a) the sides about the right angle; (b) the two oblique angles.

12. (a) Show how the solution of a quadrantal triangle may be reduced to that of a right triangle. (b) Write the relations between the sides and angles of a quadrantal triangle ABC, in which $c = 90°$.

13. In a spherical triangle ABC, $A = B$: write the relations between the sides and angles of ABC.

14. If A be one of the base angles of an isosceles spherical triangle whose vertical angle is 90° and a the opposite side, prove that $\cos a = \cot A$; and determine the limits within which it is necessary that A must lie.

15. Show how oblique spherical triangles can be solved by means of right spherical triangles. (Six cases.)

16. In a right spherical triangle ABC $(C = 90°)$ prove that: (a) $\sin^2 B - \cos^2 A = \sin^2 b \sin^2 A$; (b) $\sin A \sin 2b = \sin c \sin 2B$; (c) $\sin 2A \sin c = \sin 2a \sin B$; (d) $\sin 2a \sin 2b = 4 \cos A \cos B \sin^2 c$; (e) $\cos^2 A \sin^2 c = \sin^2 c - \sin^2 a$; (f) $\sin^2 A \cos^2 c = \sin^2 A - \sin^2 a$.

17. (a) In ABC, if $C = 90°$, and $a = b = c$, prove that $\sec A = 1 + \sec a$. (b) In ABC $(C = 90°)$ show that if $b = c = \frac{\pi}{2}$, then $\cos a = \cos A$.

18. In a right spherical triangle whose oblique angles are $72° 34'$ and $59° 42'$, find the length of the perpendicular from the right angle upon the base, and the angles which it forms with the sides.

19. Two planes intersecting at right angles are intersected by a third plane making with them angles of 60° and 75° respectively. Find the angles which the three lines of intersection make with each other.

20. Two planes intersect at right angles; from any point of their line of intersection one line is drawn in each plane making the respective angles 60° and 73° with the line of intersection. Find the angle between the two lines thus drawn.

21. A triangle whose sides are 40°, 90°, and 125° respectively, is drawn on the surface of a sphere whose radius is 8 feet. Find in feet the length of each side of this triangle, and also the angles of the polar triangle. Write the formula for finding either angle in terms of functions of the sides.

22. Solve the following spherical triangles given: (1) Right triangle, hypotenuse = 140°, one side = 20°. (2) Sides 90°, 50°, 50°. (3) Sides 100°, 50°, 60°. (4) Sides each 30° in length. (5) $A = 100°$, $C = 90°$, $a = 112°$. (6) $A = 80°$, $a = 90°$, $b = 37°$. (7) $a = b = 119°$, $C = 85°$. (8) Triangle PQR, $R = 90°$, $P = 63° 42'$, $Q = 123° 18'$. (9) Right triangle, one angle = $110° 30' 20''$, hypotenuse = $75° 45'$. (10) $A = 90°$, $b = 21° 30'$, $c = 122° 18'$. (11) $B = 90°$, $C = 79° 40'$, $b = 137° 52'$. (12) $A = 90°$, $a = 108° 23$, $c = 37° 42'$. (13) $B = 90°$, $A = 43° 10'$, $a = 78° 35'$. (14) $B = 90°$, $C = 33° 57'$, $A = 43° 18'$. (15) $A = 87° 40' 20''$, $b = 33° 42' 40''$, $B = 90°$. (16) $A = 33° 42' 40''$, $b = 87° 40' 20''$, $B = 90°$.

CHAPTER III.

1. In a spherical triangle ABC prove that: (*a*) $\sin a : \sin A = \sin b : \sin B = \sin c : \sin C$; (*b*) $\cos a = \cos b \cos c + \sin b \sin c \cos A$; (*c*) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$; (*d*) $\cos \frac{1}{2} A = \sqrt{\sin s \sin(s-a) \div \sin b \sin c}$, where $s = \frac{1}{2}(a + b + c)$; (*e*) $\tan \frac{1}{2} A \cot \frac{1}{2} B = \sin(s - b) \operatorname{cosec}(s - a)$.

2. Give the equations (or proportions) known as Napier's Analogies. Derive them.

3. Derive formulas giving the values of $\sin A$, $\cos A$, $\tan A$, and $\cos c$, in terms of functions of a, b, and c.

4. In a spherical triangle ABC show that: (*a*) If $a = b = c$, then $\sec A = 1 + \sec a$. (*b*) If $b + c = 180°$, then $\sin 2B + \sin 2C = 0$. (*c*) If $C = 90°$, then $\tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a) = \tan^2 \frac{b}{2}$.

5. In an equilateral spherical triangle show that: (*a*) $2 \sin \frac{A}{2} \cos \frac{a}{2} = 1$, and hence, that such a triangle can never have its angle less than 60°, nor its side greater than 120°; (*b*) $2 \cos A = 1 - \tan^2 \frac{a}{2}$.

6. Show that: (*a*) If the three angles of spherical triangle ABC are together equal to four right angles, then $\cos^2 \frac{c}{2} = \cot A \cot B$. (*b*) If x is

the side of a spherical triangle formed by joining the middle points of the equilateral triangle of side a, then $2 \sin \frac{x}{2} = \tan \frac{a}{2}$.

7. (a) In a spherical triangle ABC show that, if $b + c = 90°$, then $\cos a = \sin 2c \cos^2 \frac{A}{2}$. ($b$) If a be the side of an equilateral triangle and a' that of its polar triangle, prove $\cos a \cos a' = \frac{1}{2}$.

8. (a) If, in a triangle ABC, l be the length of the arc joining the middle point of the side c to the opposite vertex C, show that $\cos l = (\cos a + \cos b) \div 2 \cos \frac{c}{2}$. ($b$) In a right spherical triangle ABC ($C = 90°$), if α, β be the arcs drawn from C respectively perpendicular to and bisecting the hypotenuse c, show that $\sin^2 \frac{c}{2}(1 + \sin^2 \alpha) = \sin^2 \beta$.

9. (a) Prove that the half sum of two sides of any spherical triangle is in the same quadrant as the half sum of the opposite angles. (b) Two sides of a spherical triangle are given: prove that the angle opposite the smaller of them will be greatest when that opposite the larger is a right angle.

10. ABC is a spherical triangle of which each side is a quadrant, and P is a point within it. Prove that $\cos^2 PA + \cos^2 PB + \cos^2 PC = 1$.

11. In a spherical triangle, if $A = 36°$, $B = 60°$, and $C = 90°$, show that $a + b + c = 90°$.

CHAPTER IV.

1. (a) Name the six cases for solution of spherical triangles. (b) Discuss each case in detail, writing the formulas used in the solution, and deriving these formulas. (c) Solve an example under each case. Test the result by (1) solving by right triangles, (2) solving without logarithms, (3) using a check formula.

2. How many solutions are possible for the oblique spherical triangle ABC, given A, B, and a? Discuss in full the question of one solution, two solutions, or no solution. Plan the solution.

3. In a spherical triangle ABC, two sides a and b and the included angle C are given. Write all the formulas used in the solution and check; describe fully the process of solution. Derive the formulas used.

4. Write and deduce the formulas for finding A, B, and C of any spherical triangle when a, b, and c are given.

5. Given A, B, and C. Show how to find the remaining parts, writing the formulas to be used.

6. In an equilateral spherical triangle the side a is given. Find the angle A.

7. Solve the spherical triangle whose sides are 70°, 60°, and 50°. Solve the plane triangle obtained by connecting by straight lines the vertices of this spherical triangle, the sphere on which it is drawn being 2 feet in diameter.

8. In a triangle ABC on the earth's surface (supposed spherical) $a = 483$ miles, $b = 321$ miles, $C = 38° 21'$. Find the length of the side c. [Earth's radius = 3960 miles.]

9. Two planes intersect at an angle of 75°. From any point of their line of intersection one line is drawn in each plane, making the respective angles 55° and 80° with the line of intersection. Find the angle between the lines thus drawn.

10. Two planes intersecting at an angle of 65° are intersected by a third plane, making with them the respective angles 55° and 82°. Find the angles which the three lines of intersection make with one another.

11. A solid angle is contained by three plane angles 62°, 83°, 38°. Find the angle between the planes of the angles 62° and 38°.

12. Two of the three angles which contain a solid angle are 42° and 65° 30′, and their planes are inclined at an angle of 50°. Find the angle of the third plane face and the angles at which this third plane is inclined to the other two planes.

13. A pyramid has each of its slant sides and base an equilateral triangle. Find the angle between any two faces.

14. A pyramid each of whose slant faces is an equilateral triangle has a square base. Find the angle between any two slant faces, also the angle between any slant face and the base.

15. In the following cases ABC is a three-sided spherical figure each of whose sides is an arc of a great circle. Select those which are spherical triangles, and give reasons for so doing. Explain why the other figures cannot be triangles. Solve the triangles and check the results. (Solve some without using logarithms.)

(1) $a = 76°$, $b = 54°$, $c = 36°$. (2) $A = 54° 35' 20''$, $b = 104° 25' 45''$, $c = 92° 10'$. (3) $A = 107° 47' 7''$, $B = 38° 58' 27''$, $c = 51° 41' 14''$. (4) $A = 60°$, $B = 80°$, $C = 100°$. (5) $A = 120°$, $B = 130°$, $C = 80°$. (6) $A = 54° 35'$, $b = 104° 24'$, $c = 95° 10'$. (7) $A = 61° 37' 53''$, $B = 139° 54' 34''$, $b = 150° 17' 26''$. (8) $a = 72° 18'$, $b = 146° 35'$, $c = 98° 11'$. (9) $A = 125° 15'$, $C = 85° 12'$, $b = 100°$. (10) $A = 50°$, $B = 114° 5' 8''$, $b = 50°$. (11) $A = 83° 40'$, $b = 73° 45'$, $a = 30° 24'$. (12) $A = 83° 40'$, $b = 30° 24'$, $a = 73° 45'$. (13) $A = 97° 20'$, $a = 94° 37'$, $b = 36° 17'$. (14) $a = 127° 40'$, $b = 143° 50'$, $c = 139° 39'$. (15) $A = 40°$, $B = 30°$, $C = 20°$. (16) $A = 40° 35'$, $B = 36° 42'$, $c = 47° 18'$.

CHAPTER V.

[In the following exercises,

$$n = \sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)},$$

and $$N = \sqrt{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)};$$

also, r, r_a, r_b r_c, denote the radii of the circles inscribed in the spherical triangle ABC and its three colunar triangles, and R, R_a, R_b, R_c denote the radii of the circumscribing circles of these triangles.]

1. Given a spherical triangle ABC, find (1) the radius of the inscribed circle; (2) the radius of the circumscribing circle; (3) the radii of the inscribed circles of the colunar triangles; (4) the radii of the circumscribing circles of the colunar triangles.

Show that:

2. $\operatorname{Tan} r = \dfrac{n}{\sin s}$.

3. $\operatorname{Tan} R = \dfrac{N}{\cos(S-A)\cos(S-B)\cos(S-C)}$.

4. (*a*) $\operatorname{Cot} R \cot R_a \cot R_b \cot R_c = N^2$;

   (*b*) $\operatorname{Tan} R \cot R_a \cot R_b \cot R_c = \cos^2 S$.

5. $\operatorname{Tan} R = 4 \tan r \dfrac{-\cos S \sin s}{\sin a \sin b \sin c \sin A \sin B \sin C}$.

6. $\operatorname{Tan} r_a \tan r_b \tan r_c = \tan r \sin^2 s$.

7. $\operatorname{Tan} R + \cot r = \tan R_a + \cot r_a = \tan R_b + \cot r_b$

   $= \tan R_c + \cot r_c = \frac{1}{2}(\cot r + \cot r_a + \cot r_b + \cot r_c)$.

8. $\operatorname{Tan} R \tan r = -\dfrac{\cos S \sin a}{\sin s \sin A} = -\dfrac{\cos S \sin b}{\sin s \sin B} = \text{etc.}$ Write the other formula of this set.

9. $\operatorname{Tan}^2 R + \tan^2 R_a + \tan^2 R_b + \tan^2 R_c = \cot^2 r + \cot^2 r_a + \cot^2 r_b + \cot^2 r_c$.

10. $\operatorname{Tan} r \tan r_a \tan r_b \tan r_c = n^2$; $\cot r \tan r_a \tan r_b \tan r_c = \sin^2 s$.

11. In any equilateral triangle, $\tan R = 2 \tan r$.

12. $\operatorname{Tan} R_a = -\dfrac{\tan \frac{1}{2} a}{\cos S} = \dfrac{\cos(S-A)}{N} = \dfrac{\sin \frac{1}{2} a}{\sin A \sin \frac{1}{2} b \sin \frac{1}{2} c}$

$= \dfrac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} = \dfrac{1}{2n}[\sin s - \sin(s-a) + \sin(s-b) + \sin(s-c)]$.

Write the corresponding formulas for R_b and R_c.

13. $\cot r_a + \cot r_b + \cot r_c - \cot r = 2 \tan R$.

14. Find the radii of the circles connected with some of the triangles in Ex. 15 of the preceding set.

CHAPTER VI.

1. Define the following terms: *zone* of a sphere, *lune*, *spherical degree*, *spherical excess of a triangle*, *spherical excess of a* (non-re-entrant) *polygon*, *spherical excess* of any figure on a sphere, *spherical measure* and *spherical degree measure* of a solid angle, *spherical pyramid*, *spherical sector*, *spherical segment*.

2. Derive the area of the surface of a sphere.

3. Derive the area of a spherical triangle.

4. Discuss fully the measurement of solid angles.

5. Show how to find the spherical excess of a figure on a sphere when the area of the figure is given (in square units).

6. State and deduce *Roy's Rule* for computing the spherical excess of a triangle of known area on the earth's surface.

7. Derive the volumes of a sphere, a spherical pyramid, a spherical sector, and a spherical segment.

8. The area of an equilateral triangle is one-fourth the area of the sphere: find its sides and angles.

9. If the three sides of a spherical triangle measured on the earth's surface be 12, 16, and 18 miles, find the spherical excess.

10. If $a = b$ and $C = \frac{\pi}{2}$, show that $\tan E° = \frac{\sin^2 a}{2 \cos a}$. (In ABC.)

11. If $a = b = 60°$ and $c = 90°$, show that $E = \cos^{-1} \frac{1}{9}$. (In ABC.)

12. If $C = 90°$ in ABC, then $E = 2 \tan^{-1}(\tan \frac{1}{2} a \tan \frac{1}{2} b)$.

13. In a triangle on the earth's surface (assumed spherical), two sides are 483 and 321 miles, and the angle between them is 38° 21′. Find the area of the triangle in square miles. [Radius of earth = 3960 miles.]

14. The sides of a triangle on the earth's surface (supposed spherical) are 321, 287, and 412 miles; find the area.

15. Prove that in a right triangle ABC ($C = 90°$),

$$\cos \tfrac{1}{2} E = \frac{\cos \frac{1}{2} a \cos \frac{1}{2} b}{\cos \frac{1}{2} c}, \text{ and } \sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c}.$$

16. The spherical excess of a triangle on the earth's surface is 2″.5. Find its area, the radius of the earth being taken as 3960 miles.

17. Find the fraction of the earth's surface (supposed spherical) contained by great-circle arcs joining London, New York, and Paris. Find the spherical degree measure, and the spherical measure of the angle subtended at the centre of the earth by this part of the earth's surface.

18. Find the spherical excess of some of the triangles in Ex. 15, p. 104. Also find their areas in square inches on spheres of radii, say, 4 inches, 10 inches, 12 inches, 20 inches, a inches.

19. Find the spherical measures and the spherical degree measures of the solid angles corresponding to the triangles taken in Ex. 18.

CHAPTER VII.

1. Given the latitude and longitude of each of two places: show how to find the shortest distance between these places, and the direction of one place from the other.

2. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth: show how to find the area of the spherical triangle formed by arcs of great circles passing through them.

3. Given the sun's altitude and declination and the latitude of a place: show clearly how the time of day may be determined.

4. If d represents the sun's declination, what formulas will be required in order to determine the time of sunrise for a place whose latitude is l?

5. Show what formulas must be used to find the length of a degree of longitude on the earth's surface for a place whose latitude is l, r representing the radius of the earth.

6. The shortest distance d between two places and their latitudes l and l' are known; find their difference of longitude.

7. Given the obliquity of the ecliptic ω, and the sun's longitude λ, show that if α and δ denote his right ascension and declination respectively, then $\tan \alpha = \cos \omega \tan \lambda$, and $\sin \delta = \sin \omega \sin \lambda$.

8. The faces of a regular dodecaedron are regular pentagons, three faces meeting at each vertex. Find the diedral angle at the edge of the solid.

9. The ridges of two gable roofs meet at right angles; each roof is inclined to the horizontal at an angle of 65°. Find the diedral angle between the planes of the two roofs, and the angle their line of intersection makes with the ridge of either roof.

10. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

11. Two ports are in the same parallel of latitude, their common latitude being l, and their difference of longitude 2λ. Show that the saving of distance in sailing from one to the other on the great circle instead of sailing due east or west, is

$$2r\{\lambda \cos l - \sin^{-1}(\sin \lambda \cos l)\},$$

λ being expressed in radian measure, and r being the radius of the earth.

12. If a ship sails from New York (40° 28′ N., 74° 8′ W.) starting due east, and continues her course on an arc of a great circle, what will be her latitude when she reaches the meridian of Greenwich, and in what direction will she then be sailing?

13. Find the distance between New York (40° 28′ N., 74° 8′ W.) and Cape Clear (51° 26′ N., 9° 29′ W.), and the bearing of each from the other. [Radius of earth = 3960 miles.]

14. From Victoria, B.C. (48° 25′ N., 123° 23′ W.), a ship sails on an arc of a great circle for 1250 miles, starting in the direction S. 47° 35′ W. Find its latitude and longitude, taking the length of 1° as 69½ miles.

15. Two places are both in latitude 50° N., and the difference of their longitudes is 60°. Find the distance between them (*a*) along the parallel of latitude, (*b*) along a straight line, (*c*) along a great circle. [Earth's radius = 3960 miles.]

16. What will be the first course and the shortest (great circle) distance passed over in sailing from a place in latitude 43° N. to another place 80° east of it and in the same latitude? What is the distance between the two places along the parallel? What is the straight-line distance between them?

17. At what hours will the sun rise in London (51° 30′ 48″ N.) and New York (40° 43′ N.) when its declination is respectively 23° N., 20° N., 15° N., 10° N., 5° N., 5° S., 10° S., 15° S., 20° S., 23° S.?

18. When the sun's declination is 18°, find his right ascension and longitude.

19. What is the altitude of the sun above the horizon when its angular distance from the south point is 75° and from the west point is 60°?

20. The right ascension of Sirius is $6^h\ 38^m\ 37^s.6$, and his declination is 16° 31′ 2″ S.; the right ascension of Aldebaran is $4^h\ 27^m\ 25^s.9$, and his declination is 16° 12′ 27″ N. Find the angular distance between these stars.

21. If the sun's declination be 20° 45′ N. and his altitude be 41° 10′ at 3 P.M., find the observer's latitude.

22. What will be the altitude of the sun at 3.30 P.M. in San Francisco (37° 48′ N.), its declination being 15° S.?

23. In Bombay (18° 54′ N.) the altitude of the sun is observed to be 27° 40′. If the sun's declination is 7° S. and the observation is made in the morning, find the hour of the day.

24. Find the latitude and longitude of a star whose right ascension is $4^h\ 40^m$, and declination 57°.

25. Find the distance in degrees between the sun and moon when their right ascensions are respectively $15^h\ 12'$, $4^h\ 45'$, and their declinations are 21° 30′ S., 5° 30′ N.

26. Find the length of the longest day in the year at the following places (the sun's greatest declination being 23° 27′ N.): London (51° 30′ 48″ N.), New York (40° 43′ N.), Montreal (45° 30′ N.), St. Petersburg (60° N.), Hong Kong (22° 17′ N.).

27. Find the length of the shortest day in the year at the places mentioned in Ex. 26. (The sun's declination is then 23° 27′ S.)

28. At Copenhagen (55° 40′ N.), at an afternoon observation, the sun's altitude is 44° 20′; find the time of day, the sun's declination being 18° 25′ N.

29. At what time of day will the sun have an altitude of 53° 40′ for a place in latitude 40° 35′ N., his declination being 13° 48′ N.?

30. What will be the sun's altitude at 3.30 P.M. at a place in latitude 44° 40′ N., his declination being 18° N.?

31. What will be the sun's altitude at 10 A.M. at a place in latitude 44° 40′ N., his declination being 18° S.?

32. What is the sun's declination when his altitude at a place in latitude 37° 48′ N. is 25° at 4 P.M.?

NOTE. The Spherical Trigonometries of M'Clelland and Preston, Casey, and Bowser, contain especially good collections of exercises. See Art. 40.

ANSWERS TO THE EXAMPLES.

CHAPTER I.

Art. 24. **I. 4.** $A = 88° 12.2'$, $B = 74° 34.7'$, $C = 43° 8'$; $A = 118° 33.2'$, $B = 113° 11.2'$, $C = 92° 45'$. **II. 4.** $a = 72° 40.6'$, $b = 67° 45.8'$, $c = 51° 43.1'$; $a = 71° 22.7'$, $b = 108° 37.3'$, $c = 104° 56.7'$. **III. 4.** $A = 63° 56'$, $B = 126° 21.2'$, $c = 77° 3'$; $B = 32° 47.1'$, $C = 62° 30.7'$, $a = 84° 29.5'$. **IV. 5.** $b = 70° 5.7'$, $c = 102° 51.3'$, $A = 68° 35.8'$; $a = 46° 1.5'$, $c = 86° 0.7'$, $B = 122° 55.8'$. **VI. 3.** $B = 59° 40.1'$, $C = 114° 55'$, $c = 96° 31.1'$, and $B = 120° 19.9'$, $C = 27° 49.6'$, $c = 30° 45.4'$; $B = 65° 1.8'$, $C = 97° 16.9'$, $c = 100° 26'$; $C = 110° 43.1'$, $b = 33° 8.6'$, $c = 60° 28.8'$; $C = 165° 3.3'$, $b = 125° 1.7'$, $c = 162° 55.7'$, and $C = 119° 47'$, $c = 81° 7'$, $b = 54° 58.3'$.

CHAPTER II.

Art. 27. **4.** $c = 82° 33.9'$, $A = 60° 51.2'$, $B = 76° 56.1'$. **5.** $a = 33° 0.25'$, $b = 36° 29.4'$, $c = 47° 37.8'$.

Art. 31. **5.** (1) $C = 86° 30.9'$, $A = 36° 30.2'$, $B = 87° 25.4'$. (2) $b = 138° 24.4'$, $A = 58° 41.9'$, $B = 129° 43.1'$. (3) $a = 35° 50.6'$, $b = 75° 39.5'$, $B = 81° 29.1'$. (4) $a = 42° 49.8'$, $b = 27° 47.3'$, $c = 49° 33'$. (5) $b = 33° 37.4'$, $c = 79° 2'$, $B = 34° 20.1'$; and $b = 146° 22.6'$, $c = 100° 58'$, $B = 145° 39.9'$. (6) $a = 35° 16.4'$, $c = 51° 10.8'$, $B = 55° 18.6'$.

Art. 32. **1.** (1) $b = 54° 20'$, $A = 32° 0.75'$, $B = 57° 59.25'$, $C = 93° 59.3'$; (2) $b = 66° 29'$, $c = 111° 29.4'$, $B = 50° 17'$, $C = 128° 41.2'$. **2.** (1) $b = 59° 56.2'$, $A = 130°$, $B = 52° 55.5'$. (2) $a = 135° 33'$, $b = 100° 58.6'$, $C = 101° 24.7'$.

CHAPTER III.

Art. 37. **I. 2.** $A = 55° 58.4'$, $B = 74° 14.6'$, $C = 103° 36.6'$. **3.** $A = 43° 58'$, $B = 58° 14.4'$, $C = 108° 4.8'$. **II. 3.** $a = 39° 29.6'$, $b = 35° 36.2'$, $c = 27° 59'$. **4.** $a = 130° 49.6'$, $b = 120° 17.5'$, $c = 54° 56.1'$.

CHAPTER IV.

Art. 42. **2.** $A = 41° 27'$, $B = 66° 26.4'$, $C = 106° 3.2'$. **3.** $A = 144° 26.6'$, $B = 26° 9.1'$, $C = 36° 34.7'$.

Art. 43. **1.** $a = 43° 36'$, $b = 41° 20.9'$, $c = 33° 7.4'$. **2.** $a = 111° 40.2'$, $b = 91° 17.2'$, $c = 71° 7.4'$.

Art. 44. 2. $A = 101° 24.2'$, $B = 54° 57.9'$, $c = 79° 9.5'$. 3. $B = 78° 20.6'$, $C = 47° 47'$, $a = 82° 42'$.

Art. 45. 1. $a = 63° 15.1'$, $b = 43° 53.7'$, $C = 95° 1'$. 2. $b = 86° 39.5'$, $c = 68° 39.5'$, $A = 59° 44'$.

Art. 46. 2. $B = 36° 35.5'$, $C = 51° 59.7'$, $c = 42° 38.9'$. 3. $B = 59° 3.5'$, $C = 97° 38.8'$, $c = 56° 56.9'$; $B = 120° 56.5'$, $C = 28° 5.2'$, $c = 23° 27.8'$.

Art. 47. 1. $b = 154° 45.1'$, $c = 34° 9.1'$, $C = 70° 17.5'$. 2. $A = 164° 43.7'$, $a = 162° 37.5'$, $c = 124° 40.6'$; $A = 119° 18.7'$, $a = 81° 18.7'$, $c = 55° 19.4'$.

CHAPTER VI.

Art. 53. 1. 2827.44 sq. in. 2. 392.7 sq. in. 3. 8.25 sq. ft.

Art. 55. 1. 1.396 sq. ft. 2. 64.14 sq. ft.

Art. 56. $24° 37' 47''$ (.42986), $33° 56.6'$ (.59213), $27° 10.4'$ (.47426), $12°$ (.20944), $86° 20'$ (1.5068), etc.

Art. 57. 1. 42.986 sq. ft., 59.213 sq. ft., 47.426 sq. ft. 2. 130.9 sq. in., 941.75 sq. in.

Art. 61. 1. Spherical degree measure = 12, spherical measure = .20944, 2. Spherical degree measure = 24.63, spherical measure = .42986.

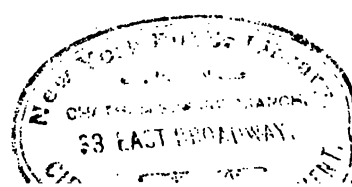
Art. 64. 1. 143.29 cu. ft., 197.38 cu. ft., 158.09 cu. ft., 1090.8 cu. in., 7847.9 cu. in., etc. 2. (*a*) 1357.17 cu. in. (*b*) 904.78 cu. ft.

CHAPTER VII.

Art. 66. 2. $8° 4.3'$ S.; course, S. $45° 6'$ E. 5. (*a*) On the equator in long. $18° 56'$ E.; course, S. $47° 39'$ E. (*b*) Lat. $42° 21'$ S., long. $108° 56'$ E.; course, E. (*c*) On the equator in long. $161° 4'$ W.; course, N. $47° 39'$ E. 6. Distance $= (51° 19.8') = 3547.675$ mi.; bearing of New York from Liverpool is N. $71° 6.8'$ W., and bearing of Liverpool from New York is N. $48° 5.8'$ E.; lat. $51° 44.1'$ N.; course, N. $65° 38'$ E.

Art. 73. 1. 8.08 A.M. 2. 2.33 P.M. 3. 2.59 P.M. 4. 4.09 P.M. 5. 6.09 P.M. 6. 9.46 A.M.

Art. 74. 1. (*a*) 5.44; (*b*) 5.06; (*c*) 4.34; (*d*) 5.43; (*e*) 6.21; (*f*) 6.53; (*g*) 7.26. 2. (*a*) 5.37; (*b*) 4.40; (*c*) 3.51; (*d*) 5.35; (*e*) 6.30; (*f*) 7.19; (*g*) 8.09. 3. (*a*) 5.29; (*b*) 4.08; (*c*) 2.51; (*d*) 5.25; (*e*) 6.42; (*f*) 7.51. (*g*) 9.09.



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